Minimal Evacuation Times and Stability

Abstract

We consider a system where packets (jobs) arrive for processing using one of the policies in a given class. We study the connection between the minimal evacuation times and the stability region of the system under the given class of policies. The result is used to establish the equality of information theoretic capacity region and system stability region for the multiuser broadcast erasure channel with feedback.

I. Introduction

In this work we consider a time slotted system where packets arrive to one of n different input queues - there may be other system queues to which packets are placed during their processing. The packets are processed by a policy from an admissible class. We study the connection between system stability and minimal *evacuation time*, i.e. the time it takes to complete processing a number of packets placed at the input queues at time 0, provided that no further arrivals occur afterwards. Under certain general assumptions on admissible policies and system statistics, it is shown that the stability region of the system is completely characterized by the asymptotic growth rate of minimal evacuation time. We make very few assumptions on the system structure and hence the result is applicable to a large number of applications in communications as well as more general control systems. However, we point out that the result, while intuitive, has to be applied with caution since there are systems for which its application leads to wrong conclusions. As an application to our methodology, we consider the N-user broadcast erasure channel with feedback. In this setup, we compare the information theoretic capacity region with the stability region and show that they are equal.

Concepts akin to evacuation time and their relation to stability have been investigated in earlier works. Baccelli and Foss [1] consider a system fed by a marked point process and operating under a given policy. The concept of *dater* is used to describe the time of last activity in the system, if the system is fed only by the mth to nth , $m \leq n$ of the points of the marked process. Assuming that the dater is a deterministic function of the arrival times and the marks of the point process, and under additional assumption on dater sample paths, they show that stability under the specified policy is characterized by the asymptotic behavior of daters. These results are extended to continuous time input processes by Altman [2]. In our setup, the system evolution may depend on random factors as well as the characteristics of the arrival process. Moreover, we do not make sample path assumptions on specific policies. We rather specify features that admissible policies may have, and based on these we characterize the stability region of the class of admissible policies by the asymptotic growth rate of minimal (over all admissible policies) evacuation times.

A different, yet related, methodology is developed by Meyn [3]; the *workload* $w(t)$ is defined as the time the server must work to clear all of the inventory of the system at time t when operating in the fluid limit. This basic concept is elaborated and used to derive significant results and obtain intuition for good control policies in specific complex networks. The concept of workload is closely related to the evacuation time, however we make minimal assumptions on system structure and the derived results are applicable to more general systems.

Regarding the relation between the information theoretic capacity and queueing theoretic stability regions, the equality of these has been shown recently in [4] for systems without feedback. The system studied in this work uses feedback, and as will be seen it can be derived in a simple manner based on stability characterization through evacuation time.

A. Preliminaries

In the following, we use the vector notation $\boldsymbol{x} = [x_1, x_2, ..., x_n]^\top$. Also $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i,\ i = 1, 2..., n$ and

$$\lceil \boldsymbol{x} \rceil \doteq [\lceil x_1 \rceil, ..., \lceil x_n \rceil],$$

where $\lceil x \rceil$ is the least integer larger than or equal to x. With $\boldsymbol{m}$, $\boldsymbol{k}$ we denote vectors with nonnegative integer coordinates and with $\boldsymbol{r}$, $\boldsymbol{s}$ vectors with nonnegative real number coordinates.

II. System Model and Admissible Policies

We consider a time-slotted system where slot $t = 0, 1, ...$ corresponds to the time interval $[t,\ t+1)$. The system has n input queues of infinite length where packets[1] arrive. Packets arriving at each input may have certain properties, e.g., service times, priorities, routing options, etc. There may be additional queues in the system, where packets may be placed during its operation. At the beginning of time slot t, i.e., at time t, $A_i(t)$ packets arrive at input i. (In particular, we use $A_i(0)$ to denote

[1]In this work we use the term *packet*, that describes an arriving unit in a communication network. However, our work applies to any general service system with arrival processes and queues, e.g. manufacturing systems, road networks, network switches, etc. Therefore, the subsequent discussion and results should be understood generically.

the number of packets in the queue of input i when the system commences operation at $t=0$.) We assume that the arrival processes satisfy the ergodicity condition

$$\lim_{t\to\infty}\frac{\sum_{\tau=0}^{t}A_i\left(\tau\right)}{t}=\lambda_i>0,\ i=1,2,...,n \tag{1}$$

as well as,

$$\lim_{t\to\infty}\frac{\mathbb{E}\left[\sum_{\tau=1}^{t}A_i\left(t\right)\right]}{t}=\lambda_i,\ i=1,2,...,n \tag{2}$$

The operation of the system is characterized by a finite set of *system states* $\mathcal{S}$, and control sets $\mathcal{G}_s$ for each $s\in\mathcal{S}$: if at the beginning of a slot the system state is $s\in\mathcal{S}$, one of the available controls $g\in\mathcal{G}_s$ is applied. There may be randomness in the behavior of the system, that is, given s and g at the beginning of a slot, the system state and the results at the end of a slot (e.g. packet erasures) may be random. For example, this makes the model particularly useful in wireless networks, where outcomes of transmissions may depend on channel state and ambient noise.

Arriving packets are processed by the system following a policy π, belonging to a class of admissible policies Π. At time t, when the system state is s, an admissible policy specifies:

1) The control $g\in\mathcal{G}_s$ to be chosen.
2) An action α among a set of available actions $\mathcal{A}_g$ when control g is chosen. An action specifies how packets are handled within the system.

The choice of controls and actions depends on the "system history" up to t, denoted by $\mathcal{H}_t$. The history $\mathcal{H}_t$ includes all information about packet arrival instants, packet departure instants, system states, controls, actions taken and results, up to and including time t.

Note that in the mathematical analysis of systems, the "state" of the mathematical model may include part of $\mathcal{H}_t$, and actions are usually not distinguished from controls. For the purposes of this work, the terms *system states* and *controls* are explicitly used to refer to the operational characteristics of the system, and are distinct from the history $\mathcal{H}_t$ and actions taken once the system characteristics are set. For example, the sizes of the queues at time t should usually be considered as part of the information captured by $\mathcal{H}_t$, rather than the system state, unless the queue states directly impact the set of controls available to the system. Also, we emphasize that the choice of one action or another within a given control (for example, which particular packet is transmitted from a given queue) does not affect the system state or slot outcome. This distinction is needed in order to define well the statistical assumptions needed for the development that follows. We next present several examples to clarify these notions.

Example 1. Assume a wireless transmitter which can transmit to a destination over one of two transmission channels, I or II (e.g. over two different carriers). Data arriving at the transmitter is classified in two types A, B. Packets from each of the classes are placed in distinct infinite size queues.

The channels can be in one of four states, $(s_1,s_2)\in\{(l,h),(h,l),(l,l),(h,h)\}$. The controls $\mathcal{G}_s$ available when in state $s=(s_1,s_2)$ determine a) the channel to be used for transmission, and b) the transmission power p. This choice determines the rate or transmission $r(p,s)$ in packets per second over the chosen channel. Once a control g is chosen, the action set $\mathcal{A}_g$ consists of two elements, a_A and a_B indicating the type of data to be transmitted over the chosen channel. The choice of action does not make a difference to the dynamics of the system state.

Example 2. Consider a communication system consisting of two nodes, a, b. Arriving packets are stored in an infinite queue at node a and must be delivered to node b. The two nodes are connected with two links, ℓ_1, ℓ_2, *at most one* of which may be activated at a time. If link ℓ_1 is activated, a packet can be successfully transmitted in one slot, but both links cannot be activated for the next 9 slots. If link ℓ_2 is activated, a transmitted packet is erased with probability .5 (and received successfully with probability .5) and both links can be activated in the next slot.

The states for this system can be described by the elements $\{0,1,2,...,9\}$, where state 0 means that both links can be activated and state $i\geq 1$ means that no link can be activated for the next i slots.

The control set for state 0 is $\mathcal{G}_0=\{g_0,\ g_1,\ g_2\}$ where g_0 means no link activation, g_1 means activation of link ℓ_1 and g_2 means activation of link ℓ_2. The control set for the rest of the states consist only of g_0. From state 0, if control g_0 or g_2 is taken, the state returns to 0 in the next slot, while if g_1 is taken the state becomes 9. From state $i\geq 1$ the system moves to state $i-1$ in the next slot.

At state 0, control g_0 results in "inactive" channels. If control g_2 is taken, the result is either "unsuccessful" or "successful" transmission on channel ℓ_2 — a random event — and if control g_1 is taken, the result is "successful transmission" on channel ℓ_1. Here, a "successful" transmission should be taken to mean that a packet will be successfully delivered to node b if transmitted in the slot (in other words, a "good" underlying transmission link); it does not preclude the respective control to include a possible action that does not make a transmission in the slot at all.

The controls under which one of the links is activated are associated with two actions: a) the action of transmitting a packet on the corresponding link, if the queue is nonempty and b) the action of not transmitting a packet ("null" action). For the control that does not activate any link, the associated action set is only the "null" action.

During system operation, there will be a number of packets at the queue of node a. The number of packets in the queue at time t is part of $\mathcal{H}_t$, not part of the system state. Based on $\mathcal{H}_t$ and s_t, a policy takes control $g \in \mathcal{G}_{s_t}$ and then an action $\alpha \in \mathcal{A}_g$. Depending on the result of the control, a transmitted packet (if any) may be successfully received, or erased.

Departures. There are well-defined times when each arriving packet is considered to depart from the system. For example, in a store-and-forward communication network where a packet arrives at node i and must be delivered to a single node j, it is natural to consider the departure time as the time at which this packet is delivered to node j. Similarly, if the packet must be multicast to a subset $\mathcal{K}$ of the nodes, the departure time of the packet can be defined as the first time at which all nodes in $\mathcal{K}$ receive the packet. However, in some systems several definitions of departure times may make sense, and the particular choice depends on the performance measures of interest. As an example, consider the case where network coding is used to transmit encoded packets. In this case, a packet p arriving at a single-destination node j may be considered as departed when the destination node j can decode the packet based on the packets already received by that node. On the other hand, if the decoded packet is still needed for decoding of other packets, it may be of interest to define the departure time of p as the first time the packet is not needed for further decoding. At any time between the arrival and departure times of a packet p, we say that p is "in the system".

There may be several restrictions on the policies in Π. We assume that all policies in Π have the following features.

Features of Admissible Policies

F1) At time t, the history of the system up to t, $\mathcal{H}_t$ is fully known.

F2) At any time t at which there are packets only at the inputs of the system, it is permissible to take controls and actions taking into account only the packet at the inputs at time t, and to proceed without taking into account the rest of history $\mathcal{H}_t$. Formally, for any time t in which the internal (non-input) queues, if any, are empty, the set of controls and actions available to a policy may only depend on the current queue state and may not depend further on $\mathcal{H}_t$.

F3) If at time t there are $\boldsymbol{k}$ packets at the inputs of the system, it is permissible to pick any $\boldsymbol{m} \leq \boldsymbol{k}$ packets and continue processing the $\boldsymbol{m}$ packets, along with other packets that may be in the system, *without taking into account* the remaining $\boldsymbol{k} - \boldsymbol{m}$ packets. Formally, the set of controls (and actions) available to a policy must be a superset of the set of controls (and actions, respectively) that would be available if $\boldsymbol{k} - \boldsymbol{m}$ packets were removed from the input queues altogether, for any $\boldsymbol{m} \leq \boldsymbol{k}$.

Features F2 and F3 may be natural for many systems, however, there are systems where they may not be available to the policies, as the following example shows.

Example 3. *Two-transmitter Aloha-type system.* Consider a system consisting of two transmitters attempting to transmit arriving packets to a single destination. Each transmitter has its own queue. Activation of both transmitters in the same slot results in loss of any packet that may be transmitted. We can model this system by considering that it has a single state, and that the control set consists of pairs (g_1, g_2) where $g_i = 1$ $(g_i = 0)$ indicates that transmitter $i \in \{1, 2\}$ becomes active (inactive).

Consider the following classes of policies, Π_1, Π_2: admissible policies π of both classes have Feature F1. Also, if only one transmitter queue, say transmitter 1 queue, has packets at time t, only the transmitter of this queue becomes active, that is the control $(1, 0)$ is chosen. However, the policies in the two classes differ when both transmitter queues are nonempty. In this case, policies in Π_1 are free to activate any of the transmitters. Under policies in Π_2 on the other hand, the controls are chosen randomly, so that each transmitter becomes active with probability q_t, $0 \leq q_t \leq 1$ (and inactive with probability $1 - q_t$), q_t being the *same* for both transmitters. An action here consists of sending a packet if a transmitter is active.

The policies in Π_1 have Feature F3, while the policies in Π_2 do not, since if, e.g. $\boldsymbol{k} = (1, 1)$, and control $(1, 1)$ is selected, packets from both queues must be transmitted at the same time, i.e, a policy is not allowed to transmit first the vector $(1, 0)$ and next the vector $(0, 1)$. Also, note that in both cases the policies trivially have feature F2.

Consider now a third class of policies, Π_3, where policies act as policies in Π_2, with the following difference: a policy $\pi \in \Pi_3$ selects again a common packet transmission probability q when both queues are nonempty; however, after a given number k of times this probability has been selected, it must thereafter remain fixed and the policy is no longer permitted to change it. For this class of policies, Feature F2 is not satisfied.

At the beginning of slot 0 let the system state be s and let there be $k_i \geq 0$, $i = 1, ..., n$ packets at input i and no arrivals afterwards, i.e., $A_i(0) = k_i$, $A_i(t) = 0$, $t = 2, 3, ...$ Let $T_s^\pi(\boldsymbol{k}) \geq 0$, $\boldsymbol{k} \neq \boldsymbol{0}$ be the time it takes until all of these packets depart from the system under policy π. We call $T_s^\pi(\boldsymbol{k})$ the *evacuation time* under policy π when the system starts in state s with $\boldsymbol{k}$ packets at the inputs, and denote its average value, $\bar{T}_s^\pi(\boldsymbol{k}) = \mathbb{E}\left[T_s^\pi(\boldsymbol{k})\right]$, $\boldsymbol{k} \neq \boldsymbol{0}$. It will also be convenient to define $\bar{T}_s^\pi(\boldsymbol{0}) = 1$, a convention that has the meaning of advancing one slot whenever the system is empty.

Let

$$\bar{T}_s^*(\boldsymbol{k}) = \inf_{\pi \in \Pi} \bar{T}_s^\pi(\boldsymbol{k}) \tag{3}$$

and

$$\bar{T}^*(\boldsymbol{k}) = \max_{s\in\mathcal{S}} \bar{T}_s^*(\boldsymbol{k}).$$

We call $\bar{T}^*(\boldsymbol{k})$ the *critical evacuation time function*. It will be seen that under certain statistical assumptions, this function determines the stability region of the policies under consideration.

Note that according to the definition of $\bar{T}_s^*(\boldsymbol{k})$, for any $\epsilon > 0$ we can always find a policy π such that

$$\bar{T}_s^{\pi}(\boldsymbol{k}) \le \bar{T}_s^*(\boldsymbol{k}) + \epsilon. \tag{4}$$

This fact will be used repeatedly in the development that follows.

Next, we present statistical assumptions regarding the system under consideration.

Statistical Assumptions

SA1) For all $\boldsymbol{k}$

$$\bar{T}_s^*(\boldsymbol{k}) \le \infty.$$

SA2) System and arrival process statistics are known to a policy.

SA3) Markings (such as service times, permissible routing paths, etc) associated with packets arriving to a given input are independent and statistically identical. Markings across inputs are independent.

SA4) If at the beginning of a slot t the system state is $s_t \in \mathcal{S}$ and control $g_t \in \mathcal{G}_{s_t}$ is taken, the results at time $t+1$ are independent of the system history before t. However, the system state s_{t+1} and the results at time $t+1$ may depend on both s_t and g_t. Hence the system states may be affected by the controls (but not actions) taken by a policy. Formally, if W_t is the (random) outcome at the end of a slot, we have for all t,

$$\Pr\left(W_{t+1}, S_{t+1}\left|s_t, g_t, \mathcal{H}_t\right.\right) = \Pr\left(W_{t+1}, S_{t+1}\left|s_t, g_t\right.\right).$$

SA5) At time $t = 0, 1, 2, \ldots$ let there be $\boldsymbol{k}$ packets in the system (where k_i is the number of packets still in the system that originally arrived at input i; they may or may not still be at the input queues). There is a policy π_h which can process all these packets until they all depart from the system by time $t + F^{\pi_h}(\boldsymbol{k})$ ($F^{\pi_h}(\boldsymbol{k})$ may be random), such that

$$\mathbb{E}\left[F^{\pi_h}(\boldsymbol{k})\right] \le C_1 \sum_{i=1}^{n} k_i + C_0, \tag{5}$$

where $C_1,\ C_0$ are finite constants (which may depend on system statistics but not on $\boldsymbol{k}$).

SA6) Let $\boldsymbol{e}_i$ be the unit n-dimensional vector with 1 at the i-th coordinate and 0 elsewhere. It holds for all $i = 1, .., n$, and all $\boldsymbol{k}$ and s,

$$\bar{T}_s^*(\boldsymbol{k}) - \bar{T}_s^*(\boldsymbol{k} + \boldsymbol{e}_i) \le D_0 < \infty. \tag{6}$$

Statistical Assumption SA5 is easy to verify in several systems. For example, in a communication network a policy that usually satisfies this assumption is the one that picks one of the $\boldsymbol{k}$ packets, transmits it to its destination, then picks another packet and so on, until all the packets are delivered to their destinations. Note that assumption SA5 implies SA1; we keep assumption SA1 separate because, as will be seen shortly, only this assumption is needed to establish the key property (namely, subadditivity) of $\bar{T}^*(\boldsymbol{k})$.

Statistical Assumption SA6 is needed to justify a technical condition in the development that follows. This assumption may also be easy to verify for several systems. It says that, if the number of packets at the system inputs at time 0 is *increased* by one, then the minimal average evacuation time under any initial state cannot be *decreased* by more than a fixed amount. For example, this assumption is always satisfied if $\bar{T}_s^*(\boldsymbol{k})$ is non-decreasing in $\boldsymbol{k}$, i.e.,

$$\bar{T}_s^*(\boldsymbol{k}) \le \bar{T}_s^*(\boldsymbol{k} + \boldsymbol{e}_i). \tag{7}$$

In particular, it can be easily shown that condition (7) holds if policies have the ability to generate "dummy" packets (i.e. packets that bear no information and are used just for policy implementation) during their operation, a feature that is available in many communication networks. Indeed, assume that at time $t = 0$ the system is in state s and there are $\boldsymbol{k}$ packets at the system inputs. Pick $\epsilon > 0$ and a policy π such that

$$\bar{T}_s^{\pi}(\boldsymbol{k} + \boldsymbol{e}_i) \le \bar{T}_s^*(\boldsymbol{k} + \boldsymbol{e}_i) + \epsilon.$$

Consider the following policy π_0 for evacuating $\boldsymbol{k}$ packets: generate a "dummy" packet for input i, place the $\boldsymbol{k} + \boldsymbol{e}_i$ packets at the inputs and use policy π to evacuate the system. By construction, $T_s^{\pi_0}(\boldsymbol{k}) \le T_s^{\pi}(\boldsymbol{k} + \boldsymbol{e}_i)$ (the inequality may be strict if the departure time of the dummy packet turns out to be strictly larger that the departure times of the rest of the packets). Hence,

$$\begin{aligned}
\bar{T}_s^*(\boldsymbol{k}) &\le \bar{T}_s^{\pi_0}(\boldsymbol{k}) \\
&\le \bar{T}_s^{\pi}(\boldsymbol{k} + \boldsymbol{e}_i) \\
&\le \bar{T}_s^*(\boldsymbol{k} + \boldsymbol{e}_i) + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, (7) follows.

To conclude the discussion of Assumption SA6, we provide an example for which Assumption SA6 holds, even though $\bar{T}_s^*(\boldsymbol{k})$ may decrease as $\boldsymbol{k}$ increases. This example is inspired by [1].

Example 4. Consider a system with two inputs. If packets from both inputs are processed simultaneously, then both depart after a slots. If a single packet from any of the inputs is processed, then this packet departs in A slots, where $A > a$. Admissible policies may select to transmit pairs of packets (one from each queue) or single packets. It is easily seen that

$$\bar{T}^*(k_1, k_2) = a \min\{k_1, k_2\} + A|k_1 - k_2|.$$

Hence, for any k, $\bar{T}^*(k, k+1) = ak + A$ and $\bar{T}^*(k+1, k+1) = a(k+1) < \bar{T}^*(k, k+1)$. On the other hand, we always have,

$$\begin{aligned}\bar{T}^*(k_1, k_2) - \bar{T}^*(k_1+1, k_2) &= a\min\{k_1, k_2\} + A|k_1 - k_2| - a\min\{k_1+1, k_2\} - A|k_1 + 1 - k_2| \\ &\le A.\end{aligned}$$

III. Properties of Critical Evacuation Time Function

The following property of the critical evacuation time function will play a key role in the following.

Lemma 5. *The Critical Evacuation Time Function is subadditive, i.e., the following holds for* $\boldsymbol{m} \ge \boldsymbol{0}$, $\boldsymbol{k} \ge \boldsymbol{0}$

$$\bar{T}^*(\boldsymbol{k} + \boldsymbol{m}) \le \bar{T}^*(\boldsymbol{k}) + \bar{T}^*(\boldsymbol{m}) \tag{8}$$

Proof: Let $\epsilon > 0$ and let the system be in state s at time 0. An admissible policy π that evacuates $\boldsymbol{k} + \boldsymbol{m}$ packets is the following.

a) Pick an admissible policy $\pi_{\boldsymbol{k}}$ such that,

$$\bar{T}_s^{\pi_{\boldsymbol{k}}}(\boldsymbol{k}) \le \bar{T}_s^*(\boldsymbol{k}) + \epsilon/2.$$

b) Evacuate the $\boldsymbol{k}$ packets following policy $\pi_{\boldsymbol{k}}$. According to Feature F3 this is permissible. From Statistical Assumption SA4 we conclude that the average evacuation time in this case is $\bar{T}_s^{\pi_{\boldsymbol{k}}}(\boldsymbol{k})$. Let s_1 be the state of the system by time $T_s^{\pi_{\boldsymbol{k}}}(\boldsymbol{k})$. Both s_1 and $T_s^{\pi_{\boldsymbol{k}}}(\boldsymbol{k})$ are known to $\pi_{\boldsymbol{k}}$ (hence to π), due to Feature F1. Note that s_1 is a random variable that depends on s.

c) Again, pick an admissible policy $\pi_{\boldsymbol{m}}$ such that,

$$\bar{T}_{s_1}^{\pi_{\boldsymbol{m}}}(\boldsymbol{m}) \le \bar{T}_{s_1}^*(\boldsymbol{m}) + \epsilon/2,$$

According to Feature F2, this choice of $\pi_{\boldsymbol{m}}$ is permissible.

d) Evacuate the $\boldsymbol{m}$ packets following policy $\pi_{\boldsymbol{m}}$. Due to Statistical Assumption SA3 and SA4, the average evacuation time (given s_1) in this case is $\bar{T}_{s_1}^{\pi_{\boldsymbol{m}}}(\boldsymbol{m})$.

The average evacuation time of π is

$$\begin{aligned}\bar{T}_s^{\pi}(\boldsymbol{k} + \boldsymbol{m}) &= \bar{T}_s^{\pi_{\boldsymbol{k}}}(\boldsymbol{k}) + \mathbb{E}\left[\bar{T}_{s_1}^{\pi_{\boldsymbol{m}}}(\boldsymbol{m})\right] \\ &\le \bar{T}_s^*(\boldsymbol{k}) + \mathbb{E}\left[\bar{T}_{s_1}^*(\boldsymbol{m})\right] + \epsilon,\end{aligned} \tag{9}$$

where the expectation in (9) is with respect to random variable s_1. Hence,

$$\begin{aligned}\bar{T}^*(\boldsymbol{k} + \boldsymbol{m}) &= \max_{s\in\mathcal{S}} \bar{T}_s^*(\boldsymbol{k} + \boldsymbol{m}) \\ &\le \max_{s\in\mathcal{S}} \bar{T}_s^{\pi}(\boldsymbol{k} + \boldsymbol{m}) \quad \text{according to (3)} \\ &\le \max_{s\in\mathcal{S}} \left\{\bar{T}_s^*(\boldsymbol{k}) + \mathbb{E}\left[\bar{T}_{s_1}^*(\boldsymbol{m})\right]\right\} + \epsilon \quad \text{according to (9)} \\ &\le \max_{s\in\mathcal{S}} \bar{T}_s^*(\boldsymbol{k}) + \max_{s\in\mathcal{S}} \bar{T}_s^*(\boldsymbol{m}) + \epsilon \quad \text{since } \bar{T}_{s_1} \le \max_{s\in\mathcal{S}} \bar{T}_s\end{aligned}$$

Since ϵ is arbitrary, the lemma follows. ∎

Let $\mathbb{N}_0$ and $\mathbb{R}_0$ be respectively the set of nonnegative integers and nonnegative real numbers. We extend the domain of definition of $\bar{T}^*(\boldsymbol{k})$ from $\mathbb{N}_0^n$ to $\mathbb{R}_0^n$ as follows. For $\boldsymbol{r} \in \mathbb{R}_0^n$, let

$$\bar{T}^*(\boldsymbol{r}) = \bar{T}^*(\lceil \boldsymbol{r} \rceil). \tag{10}$$

The function $\bar{T}^*(\boldsymbol{r})$ is not necessarily subadditive in $\mathbb{R}_0^n$, since, in general, subadditivity at integer points does not imply subadditivity over $\mathbb{R}_0$. For example, the function $f(2l) = al$ and $f(2l+1) = al + A$, $l = 0, 1, ...$, with $a < A$, is subadditive in $\mathbb{N}_0$, while for $r_1 = r_2 = 1.5$, $f(\lceil r_1 + r_2 \rceil) = f(3) = a + A$ and $f(\lceil r_1 \rceil) + f(\lceil r_2 \rceil) = 2a < f(\lceil r_1 + r_2 \rceil)$. However, as the next Lemma shows, $\bar{T}^*(\boldsymbol{r})$ possesses the basic property of subadditive functions, namely the asymptotically linear rate of growth.

Theorem 6. *For any $\boldsymbol{r} \in \mathbb{R}_0^n$, the limit function*

$$\hat{T}(\boldsymbol{r}) = \lim_{t\to\infty} \frac{\bar{T}^*(t\boldsymbol{r})}{t}, \tag{11}$$

exists and is finite, positively homogeneous, convex and Lipschitz continuous, i.e., it holds

$$\left|\hat{T}(\boldsymbol{r}) - \hat{T}(\boldsymbol{s})\right| \leq D\sum_{i=1}^{n} |r_i - s_i|,$$

where D is a positive constant. Moreover, for any sequence $\boldsymbol{r}_t \in \mathbb{R}_0^n$ such that

$$\lim_{t\to\infty} \boldsymbol{r}_t = \boldsymbol{\lambda} < \boldsymbol{\infty},$$

it holds

$$\lim_{t\to\infty} \frac{\bar{T}^*(t\boldsymbol{r}_t)}{t} = \hat{T}(\boldsymbol{\lambda}). \tag{12}$$

Here, "positively homogeneous" means that for any $\rho \geq 0$,

$$\hat{T}(\rho\boldsymbol{r}) = \rho\hat{T}(\boldsymbol{r}). \tag{13}$$

The proof of Theorem 6 is given in the Appendix.

IV. Stability - Necessity

Let $D_{s,i}^{\pi}(t)$, $t \geq 1$, be the number of packet arrivals at input i that have departed from the system during time slot t under policy $\pi \in \Pi$ when the system starts in state s. Define also $D_{s,i}^{\pi}(0) = 0$. In the following we will use the notation

$$\tilde{A}_i(t) = \sum_{\tau=0}^{t} A_i(\tau), \quad \tilde{D}_{s,i}^{\pi}(t) = \sum_{\tau=0}^{t} D_{s,i}^{\pi}(\tau),$$

to denote the cumulative number or arrivals and departures respectively up to time t. Hence the number of packet arrivals at input i that are still in the system at time t is $Q_{s,i}^{\pi}(t) = \tilde{A}_i(t) - \tilde{D}_{s,i}^{\pi}(t)$ (these packets may at time t be scattered among internal system queues as well as the original input queue). We define the vector $\boldsymbol{Q}_s^{\pi}(t) = \left(Q_{s,i}^{\pi}(t)\right)_{i=1}^{n}$ and the total system occupancy

$$Q_s^{\pi}(t) = \sum_{i=1}^{n} Q_{s,i}^{\pi}(t).$$

Let $\mathcal{M}$ be a probability measure over the space of permissible arrival processes; in other words, $\mathcal{M}$ captures the statistical assumptions about the arrival processes, such as the distribution of the arrival sizes, whether or not the arrivals are independent over time and between queues, etc. Let $\mathcal{M}_{\boldsymbol{\lambda}}$ be a probability measure over arrival processes that satisfy ergodicity conditions (1)–(2) with a rate vector $\boldsymbol{\lambda}$.

Definition 7. ***System Stability.*** A policy $\pi \in \Pi$ is called stable for an arrival rate vector $\boldsymbol{\lambda} \geq \boldsymbol{0}$, if under any initial system state s, the following holds:

$$\lim_{q\to\infty} \limsup_{t\to\infty} \Pr\left(Q_s^{\pi}(t) > q\right) = 0 \tag{14}$$

(where the probability in (14) is taken with respect to the arrival process statistics $\mathcal{M}_{\boldsymbol{\lambda}}$, as well as the system internal state transitions).

The stability region $\mathcal{R}^{\pi}$ of a policy π (under $\mathcal{M}$) is the closure of the set of the arrival rate vectors for which the policy is stable. The stability region $\mathcal{R}$ of the system is the closure of the union of $\mathcal{R}^{\pi}, \pi \in \Pi$. [2]

We show in Theorem 9 below that under (1) and (2), it holds $\mathcal{R} \subseteq \left\{\boldsymbol{r} \geq \boldsymbol{0} : \hat{T}(\boldsymbol{r}) \leq 1\right\}$. Furthermore, in section V we show that under the assumption that the packet arrival vectors are i.i.d. over time, we also have $\left\{\boldsymbol{r} \geq 0 : \hat{T}(\boldsymbol{r}) \leq 1\right\} \subseteq \mathcal{R}$, hence, $\mathcal{R} = \left\{\boldsymbol{r} \geq 0 : \hat{T}(\boldsymbol{r}) \leq 1\right\}$, and we show an explicit policy called "Epoch-based" that is stabilizing.

For the proof of Theorem 9 we need the following lemma.

[2] We emphasize that the stability region of a policy may in general depend on the permitted statistical assumptions about the arrival processes; for example, a policy may be unstable for a certain rate vector $\boldsymbol{\lambda}$ if general stationary arrival processes are allowed, but become stable if the individual queue arrivals are required to be independent. The above definition of stability is generic and captures a number of common definitions of stability in the literature, and the subsequent discussion in this section is orthogonal to any specific assumptions imposed on the arrival process, beyond the basic ergodicity condition of (1)–(2).

Lemma 8. *If (14), (1), (2) hold, then*

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[Q_s^\pi(t)\right]}{t} = 0. \tag{15}$$

Proof: It follows from (1), (2) and the corollary to Theorem 16.14 in [5] that the sequences $\left\{\tilde{A}_i(t)/t\right\}$ $i=1,..,n$ are uniformly integrable, hence the sequence $\left\{\sum_{i=1}^n \tilde{A}_i(t)/t\right\}$ is also uniformly integrable. Since

$$0 \le \frac{Q_s^\pi(t)}{t} \le \frac{\sum_{i=1}^n \tilde{A}_i(t)}{t},$$

we conclude that the sequence $\{Q_s^\pi(t)/t\}$ is also uniformly integrable. We will show in the next paragraph that $\{Q_s^\pi(t)/t\}$ converges in probability to 0. Equality (15) will then follow from Theorem 25.12 in [5].

Pick any $\epsilon > 0$ (arbitrarily small) and a $q \ge 0$ large enough so that according to (14) it holds,

$$\limsup_{t\to\infty} \Pr\{Q_s^\pi(t) > q\} \le \epsilon$$

Since we can pick t_0 large enough so that $\epsilon t > q$, $t \ge t_0$, we have

$$\begin{aligned}\limsup_{t\to\infty} \Pr\left\{\frac{Q_s^\pi(t)}{t} > \epsilon\right\} &\le \limsup_{t\to\infty} \Pr\{Q_s^\pi(t) > q\}\\ &\le \epsilon\end{aligned}$$

i.e., $\{Q_s^\pi(t)/t\}$ converges in probability to 0. ■

Theorem 9. *Let (1), (2) hold. If $\boldsymbol{r} \in \mathcal{R}$ then,*

$$\hat{T}(\boldsymbol{r}) \le 1.$$

Proof: Pick $\boldsymbol{r} \in \mathcal{R}$. Since $\boldsymbol{r}$ belongs to the closure of the rates for which the system is stabilizable, for any $\delta > 0$ we can find a $\boldsymbol{\lambda} \ge \mathbf{0}$, $\|\boldsymbol{\lambda} - \boldsymbol{r}\| \le \delta$, for which the system is stable under some policy $\pi_0 \in \Pi$. By continuity of $\hat{T}(\boldsymbol{r})$ it suffices to show that for any such $\boldsymbol{\lambda}$,

$$\hat{T}(\boldsymbol{\lambda}) \le 1. \tag{16}$$

Let the initial system state be $s \in \mathcal{S}$. Fix an arbitrary time index t and generate random number of packets $\boldsymbol{A}(0), ..., \boldsymbol{A}(t)$ according to the distribution of the arrival processes. Consider that all $\tilde{\boldsymbol{A}}(t) = \sum_{\tau=0}^t \boldsymbol{A}(t)$ packets are in the system at the beginning of time and construct the following evacuation policy π.

1. Mimic the actions of policy π_0 for up to t time slots, assuming that the packet arrival process at time τ is $\boldsymbol{A}(\tau)$, $\tau = 1, ..., t$. Due to Statistical Assumption SA2 and Features F1, F3 this mimicking is permissible.[3]

2. If all $\tilde{\boldsymbol{A}}(t)$ packets are transmitted by time t then the evacuation time of π is at most t. Else, after t time slots there will be $\boldsymbol{Q}_s^{\pi_0}(t) > 0$ packets in the system. According to Statistical Assumption SA5, pick a policy π_h to evacuate the $\boldsymbol{Q}_s^{\pi_0}(t)$ packets in $F^{\pi_h}(\boldsymbol{Q}_s^{\pi_0}(t))$ slots, where

$$\mathbb{E}\left[F^{\pi_h}(\boldsymbol{Q}_s^{\pi_0}(t)) \middle| \tilde{\boldsymbol{A}}(t), \tilde{\boldsymbol{D}}_s^{\pi_0}(t)\right] \le C_1 Q_s^{\pi_0}(t) + C_0, \qquad \text{by (5).} \tag{17}$$

The evacuation time of π given $\tilde{\boldsymbol{A}}(t)$ is at most $t + F^{\pi_h}(\boldsymbol{Q}_s^{\pi_0}(t))$ — "at most", because all $\tilde{\boldsymbol{A}}(t)$ packets may have left before time t — and hence, taking the conditional average, we have

$$\begin{aligned}\bar{T}_s^*\left(\tilde{\boldsymbol{A}}(t)\right) &\le t + \mathbb{E}\left[F^{\pi_h}(\boldsymbol{Q}_s^{\pi_0}(t)) \middle| \tilde{\boldsymbol{A}}(t)\right]\\ &= t + \mathbb{E}\left[\mathbb{E}\left[F^{\pi_h}(\boldsymbol{Q}_s^{\pi_0}(t)) \middle| \tilde{\boldsymbol{A}}(t), \tilde{\boldsymbol{D}}_s^{\pi_0}(t)\right] \middle| \tilde{\boldsymbol{A}}(t)\right]\\ &\le t + C_1 \mathbb{E}\left[Q_s^{\pi_0}(t) \middle| \tilde{\boldsymbol{A}}(t)\right] + C_0 \text{ by (17)}\end{aligned}$$

Next, using the last inequality,

$$\begin{aligned}\bar{T}^*\left(\tilde{\boldsymbol{A}}(t)\right) &= \max_{s\in\mathcal{S}}\left\{\bar{T}_s^*\left(\tilde{\boldsymbol{A}}(t)\right)\right\}\\ &\le t + C_1 \max_{s\in\mathcal{S}} \mathbb{E}\left[Q_s^{\pi_0}(t) \middle| \tilde{\boldsymbol{A}}(t)\right] + C_0\\ &\le t + C_1 \sum_{s\in\mathcal{S}} \mathbb{E}\left[Q_s^{\pi_0}(t) \middle| \tilde{\boldsymbol{A}}(t)\right] + C_0.\end{aligned}$$

[3] We remark that the theorem continues to hold even if *anticipative* policies are allowed, i.e., if Feature F1 is revised so that the information available to a policy includes not just the past history up to time t, but future packet arrivals as well. If π_0 is anticipative, one can accordingly generate random variables $\boldsymbol{A}(\tau)$, $\tau = t+1, \ldots$ so that π can mimic π_0 taking into account the future arrivals; the rest of the proof then remains unchanged.

Taking expectations with respect to $\tilde{\boldsymbol{A}}(t)$ and dividing by t, we have

$$\mathbb{E}\left[\frac{\bar{T}^*\left(\frac{\tilde{\boldsymbol{A}}(t)}{t}t\right)}{t}\right] \le 1 + C_1\sum_{s\in\mathcal{S}}\frac{\mathbb{E}\left[Q_s^{\pi_0}(t)\right]}{t} + \frac{C_0}{t} \tag{18}$$

Since

$$\lim_{t\to\infty}\frac{\tilde{\boldsymbol{A}}(t)}{t} = \boldsymbol{\lambda}, \text{ by } (1),$$

using (12) from Theorem 6 we then obtain,

$$\lim_{t\to\infty}\frac{\bar{T}^*\left(\frac{\tilde{\boldsymbol{A}}(t)}{t}t\right)}{t} = \hat{T}(\boldsymbol{\lambda})$$

Hence,

$$\begin{aligned}
\hat{T}(\boldsymbol{\lambda}) &= \mathbb{E}\left[\lim_{t\to\infty}\frac{\bar{T}^*\left(\frac{\tilde{\boldsymbol{A}}(t)}{t}t\right)}{t}\right] \\
&\le \liminf_{t\to\infty}\mathbb{E}\left[\frac{\bar{T}^*\left(\frac{\tilde{\boldsymbol{A}}(t)}{t}t\right)}{t}\right] \quad \text{by Fatou's lemma} \\
&\le \liminf_{t\to\infty}\left(1 + C_1\sum_{s\in\mathcal{S}}\frac{\mathbb{E}\left[Q_s^{\pi_0}(t)\right]}{t} + \frac{C_0}{t}\right) \quad \text{by } (18) \\
&= 1 + C_1\sum_{s\in\mathcal{S}}\lim_{t\to\infty}\frac{\mathbb{E}\left[Q_s^{\pi_0}(t)\right]}{t} + \lim_{t\to\infty}\frac{C_0}{t} \text{ by } (15) \\
&= 1 \text{ by } (15)
\end{aligned}$$

■

We note that there are classes of policies for which the limit $\hat{T}(\boldsymbol{\lambda})$ can be formally defined, but Theorem 9 does not hold in all its generality since some of the Features of admissible policies in Section II are not satisfied. The next example shows the case where Feature F3 is not satisfied.

Example 10. Consider the following system. There are two inputs. Policies may decide to process no packets in a slot, otherwise processing of packets must obey the following rule. If only one of the inputs has packets a single packet from the nonempty input is processed in 1 time slot. If on the other hand both queues are nonempty, then pairs of packets from both queues must be processed in 3 time slots. This system is a simplified version of the system in Example 3 and the specified policies do not satisfy Feature F3. It can be easily seen that

$$\bar{T}^*(k_1,k_2) = 3\min\{k_1,k_2\} + |k_1 - k_2|,$$

hence formally,

$$\hat{T}(r_1,r_2) = 3\min\{r_1,r_2\} + |r_1 - r_2|.$$

The region $\hat{T}(r_1,r_2) \le 1$ is described by

$$\{\boldsymbol{r}\ge\boldsymbol{0} : r_1 + 2r_2 \le 1, \text{ and } r_1 \ge r_2\} \cup \{\boldsymbol{r}\ge\boldsymbol{0} : 2r_1 + r_2 \le 1, \text{ and } r_2 \ge r_1\} \tag{19}$$

Clearly, the vector $(1/2,1/2)$does not belong in this region. Consider, however that 1 packet arrives in even slots to input 1 and 1 packet in odd slots to input 2, hence the arrival rate vector is $(1/2,1/2)$. Then simply processing immediately the arriving packets results in a stable policy.

Notice also that the region in (19) is not convex, while the region in Theorem 9 is convex since $\hat{T}(r_1,r_2)$ is convex.

The arrival processes in the previous example are not stationary, hence one may wonder whether imposing slightly stronger assumptions on the arrival processes would render the claim of Theorem 9 valid. An example is presented below, where the arrival processes are i.i.d. but Theorem 9 still does not hold since admissible policies do not satisfy Feature F3.

Example 11. Let $M > 1$ and consider a system with a single input and the following restriction on the policies. If the number of packets in the inputs is

$$k = lM + \upsilon,\ 0 \le \upsilon \le M-1,$$

then a policy may either decide to idle in a slot or to transmit m packets, $1 \leq m \leq M + \upsilon$ in which case it takes l slots to process all m packets. Under this restriction we have

$$\begin{aligned}\bar{T}^*(k) &= \sum_{i=1}^{l} i,\\ &= \frac{l(l+1)}{2}\end{aligned}$$

hence,

$$\begin{aligned}\hat{T}(r) &= \lim_{t\to\infty} \frac{\bar{T}^*(\lceil tr \rceil)}{t}\\ &= \lim_{t\to\infty} \frac{1}{t}\left(\frac{(\lceil tr \rceil - \upsilon_t)(\lceil tr \rceil - \upsilon_t + 1)}{2M^2}\right)\\ &= \infty.\end{aligned}$$

Applying formally Theorem 9 we deduce that the system is unstable for any positive arrival rate. Consider, however, that the arrival process is i.i.d but bounded, such that at most $2M-1$ packets may arrive at the beginning of each slot (including slot 0, i.e. to be in the system when it commences operation). Then the policy that transmits all the packets immediately is stable, i.e., under the stated conditions on arrival process statistics, the system is stable for any arrival rate $\lambda \leq 2M-1$.

For the systems described in the last two examples, there were rates outside the region obtained by using formally $\hat{T}(\boldsymbol{r})$, for which the systems were stabilizable. The next example shows an opposite case, namely where the system is unstable for rates inside the formally obtained region (again, due to not satisfying Feature F3).

Example 12. *System with priorities and switchover times.* Consider a single server with two inputs, where arrivals at input 1 have priority over arrivals at input 2. If there are packets from input 1 in the system, one of these packets must be served. Packets from input 2 may be delayed by a policy. Packets are of length 1 slot. There is a preparatory time of 1 slot to set the system to serve packets from a given input. Hence, when the system changes from serving packets of one input to serving packets of the other input, there is an idle slot. The system may start by having the server ready to serve one of the two inputs.

The system has two states, $s_1,\ s_2$, where state s_i means that the server is set to serve packets of input i. For this system, we have

$$\bar{T}^*_{s_1}(k_1,k_2) = \begin{cases} k_1 + 1 + k_2 & \text{if } k_2 \neq 0\\ k_1 & \text{if } k_2 = 0\end{cases}$$

and

$$\bar{T}^*_{s_2}(k_1,k_2) = \begin{cases} 1 + k_1 + 1 + k_2 & \text{if } k_1 \neq 0,\ k_2 \neq 0\\ 1 + k_1 & \text{if } k_2 = 0\\ k_2 & \text{if } k_1 = 0\end{cases}$$

Hence,

$$\hat{T}(r_1,r_2) = r_1 + r_2$$

and the region obtained formally is

$$\{\boldsymbol{r} \geq \boldsymbol{0} :\ r_1 + r_2 \leq 1\}.$$

Consider, however an arrival pattern where the system starts at state s_1, and a single packet arrives at input 1 at every $t = 4k,\ k = 0, 1, ...$; hence $\lambda_1 = .25$. Packets at input 2 arrive according to an i.i.d process of rate $\lambda_2 > .5$. It can be easily checked that in any interval $[4k,\ 4k+8)$, the number of packets served from input 2 cannot be larger than 4, hence the departure rate for packet at input 2 cannot be more than .5 and the system is unstable, even though $\lambda_1 + \lambda_2 < 1$.

One may wonder whether if the initial state of the system at time $t = 0$ is fixed, say $s(0) = s_0$, then stability is determined by $\bar{T}^*_{s_0}(\boldsymbol{k})$ only. The following final example illustrates that this is not always the case, i.e. the condition of theorem 9 applies to the critical (worst-case) evacuation time function, and not just the evacuation time function corresponding to s_0.

Example 13. Consider a system with two servers, where server 1 takes l slots to serve a packet, and server 2 takes $L > l$ slots. The system can be in one of three states, (0,0), (1,0), (0,1), where 0 denotes an inactive and 1 denotes an active server. Suppose that there are no (or null) controls, and that state transitions are random with the following transition probabilities.

$$\Pr\{(1,0)\,|(0,0)\} = \Pr\{(0,1)\,|(0,0)\} = \Pr\{(0,0)\,|(0,0)\} = \frac{1}{3},\ \Pr\{(1,0)\,|(1,0)\} = \Pr\{(0,1)\,|(0,1)\} = 1.$$

If the system starts at state $(0,0)$, it takes on average 1.5 slots to move to one of the other states, and the transition to either state occurs with equal probability. Then, since no further change of states occurs afterwards, it will take either lk or Lk slots to evacuate k packets. Hence,

$$\bar{T}^{*}_{(0,0)}\left(k\right)=\frac{3}{2}+\frac{l+L}{2}k.$$

It can also be easily verified that

$$\bar{T}^{*}_{(1,0)}\left(k\right)=lk$$
$$\bar{T}^{*}_{(0,1)}\left(k\right)=Lk$$

hence,

$$\bar{T}^{*}\left(k\right)=\max\left\{\frac{3}{2}+\frac{l+L}{2}k,\ lk,\ Lk\right\}$$

and $\hat{T}\left(r\right)=Lr$, which results in the stability condition, $\lambda\leq\frac{1}{L}$.

Assume now that the system starts in state $s=(0,0)$ (an initial state that may be "natural" in some sense), and formally use $\bar{T}^{*}_{(0,0)}\left(k\right)$ in place of $\bar{T}^{*}\left(k\right)$. Then, we would conclude that $\hat{T}\left(r\right)=\frac{l+L}{2}r$ and hence that the system is stable when

$$\lambda\leq\frac{2}{l+L}.$$

This, however, is wrong since for $\frac{2}{l+L}>\lambda>\frac{1}{L}$, under state transition $(0,0)\rightarrow(0,1)$, an event of positive probability, the input rate will be larger than the output rate.

V. Epoch Based Policy - Sufficiency

In this section, we consider a specific policy which we henceforth refer to as an *Epoch-Based* policy. The idea of the policy (which is defined formally below) is to divide the time into 'epochs' and focus on the efficient evacuation of packets present in the system at the start of an epoch, while ignoring any new packets that arrive during the epoch. The main result of this section is that, for the special case of independent and identically distributed (i.i.d) arrival processes, the epoch-based policy is throughput-optimal, provided that the underlying evacuation policy within each epoch is efficient (i.e., informally, minimizes the expected evacuation time for the packets present at the start of the epoch). More precisely, in this section we make the assumption that the arrival process vectors $\boldsymbol{A}(t)$ are i.i.d with respect to time for $t=1,2,\dots$ (for a given time slot t, the components of the vector $\boldsymbol{A}(t)$ may be dependent; also, the initial number of packets in the system at $t=0$, namely $\boldsymbol{A}(0)$, can be arbitrary and is not required to have the same distribution as for $t\geq1$). We then show that the epoch-based policy is stabilizing for any such arrival processes if the arrival rate $\boldsymbol{\lambda}$ satisfies $\hat{T}\left(\boldsymbol{\lambda}\right)<1$.

Consider the set

$$\mathcal{R}_{l}=\left\{\boldsymbol{\lambda}\geq\mathbf{0}:\ \hat{T}\left(\boldsymbol{\lambda}\right)<1\right\}.$$

This set is nonempty, since

$$\hat{T}\left(\mathbf{0}\right)=\lim_{t\rightarrow\infty}\frac{\bar{T}^{*}\left(t\cdot\mathbf{0}\right)}{t}=0,\tag{20}$$

hence $\mathbf{0}\in\mathcal{R}_{l}$. We will construct a policy that is stable for any $\boldsymbol{\lambda}\in\mathcal{R}_{l}$. The continuity, convexity of $\hat{T}\left(\boldsymbol{\lambda}\right)$ and (20) imply that the closure of $\mathcal{R}_{l}$ is the set $\left\{\boldsymbol{\lambda}\geq\mathbf{0}:\ \hat{T}\left(\boldsymbol{\lambda}\right)\leq1\right\}$ and hence,

$$\left\{\boldsymbol{\lambda}\geq\mathbf{0}:\hat{T}\left(\boldsymbol{\lambda}\right)\leq1\right\}\subseteq\mathcal{R}.$$

Combined with the necessity result of Section IV, we then conclude that

$$\mathcal{R}=\left\{\boldsymbol{\lambda}\geq\mathbf{0}:\hat{T}\left(\boldsymbol{\lambda}\right)\leq1\right\}.$$

We now present a policy that stabilizes the system for any $\boldsymbol{\lambda}\in\mathcal{R}_{l}$, that is,

$$\hat{T}\left(\boldsymbol{\lambda}\right)<1.\tag{21}$$

A version of this policy was used in [6] to provide a stabilizing policy for a two-user broadcast erasure channel with feedback.

Definition 14. ***Epoch-Based Policy*** π_{ϵ}: Pick $\epsilon>0$ such that

$$0<\epsilon<1-\hat{T}\left(\boldsymbol{\lambda}\right),$$

and for each $\boldsymbol{k}$ and s, pick an evacuation policy $\pi_{\boldsymbol{k},s}$ such that

$$\bar{T}^{\pi_{\boldsymbol{k},s}}_{s}\left(\boldsymbol{k}\right)\leq\bar{T}^{*}_{s}\left(\boldsymbol{k}\right)+\epsilon.\tag{22}$$

Policy π_ϵ operates recursively in (random) time intervals $[t_{m-1}, t_m)$, $m = 1, ...$, called “epochs”, as follows. Epoch 1 starts at time $t_0 = 0$ at state $S_0 = s_0$ with $\tilde{\boldsymbol{A}}(0) = \boldsymbol{A}(0) = \boldsymbol{k}$ packets at the inputs; policy $\pi_{\boldsymbol{k},\boldsymbol{s}}$ is used to evacuate the $\boldsymbol{k}$ packets by time $t_1 = T_s^{\pi_{\boldsymbol{k},\boldsymbol{s}}}(\boldsymbol{k})$, while any new packet arrivals during the epoch are kept at the inputs, but excluded from processing. Let S_m be the state of the system at time t_m. Epoch $m+1$, $m \geq 1$ starts at time t_m with $\boldsymbol{k}_m = \tilde{\boldsymbol{A}}(t_m) - \tilde{\boldsymbol{A}}(t_{m-1})$ packets at the inputs and policy $\pi_{\boldsymbol{k}_m, S_m}$ is used to evacuate the $\boldsymbol{k}_m$ packets by time t_{m+1}.

Let $T_m = t_m - t_{m-1}, m = 1, 2, \ldots$ be the length of the m-th epoch. Since the arrival process vector is i.i.d, if policies satisfy the Basic Features and the Statistical Assumptions of Section II, the process $\{(T_m, S_m)\}_{m=1}^{\infty}$ constitutes a (homogeneous) Markov chain with stationary transition probabilities. Note that with this formulation, the initial state of the Markov chain, (T_1, S_1), is a random variable whose distribution depends on $\boldsymbol{A}(0)$ and s_0.

The main result of this section is the following.

Theorem 15. *For any $\boldsymbol{\lambda} \geq \mathbf{0}$ such that*

$$\hat{T}(\boldsymbol{\lambda}) < 1, \tag{23}$$

policy π_ϵ stabilizes the system.

The proof of this theorem is given in the Appendix.

Remarks

1) The epoch-based policy is non-anticipative (it does not require knowledge of future packet arrivals), but is sufficient to attain the stability region even if anticipative policies are allowed, as explained in the footnote in the proof of Theorem 9. Thus, the ability to anticipate future packet arrivals is not required for throughput optimality.
2) Note that stability depends solely on the fact that inequality (22) holds for large enough $|\boldsymbol{k}|$. Hence, for the epoch based policy to be stable, it is sufficient for policies $\pi_{\boldsymbol{k},\boldsymbol{s}}$ to satisfy (22) only for large enough $|\boldsymbol{k}|$. In other words, asymptotically optimal evacuation policies can be used to construct stabilizing epoch based policies.
3) The requirement for the arrival process to be i.i.d. only applies for $t \geq 1$; the initial queue lengths, namely $\boldsymbol{A}(0)$, may have any distribution that is not necessarily the same as for $\boldsymbol{A}(t), t = 1, 2, \ldots$. By induction, it is easy to extend the “exemption” up to any finite t_0 and only require the arrival process to be i.i.d. for $t > t_0$.
4) Similarly, the proof can be easily extended to the case where the arrival process is not i.i.d for individual time slots but is “block-i.i.d” with block length D; in other words, where the vectors $\langle \boldsymbol{A}(i \cdot D + 1), \ldots, \boldsymbol{A}((i+1) \cdot D) \rangle$ are i.i.d with respect to i for $i = 1, 2, \ldots$ (but arrivals may be interdependent within a “time block” $iD + 1 \leq t \leq (i+1)D$). This is achieved via time scaling by a factor of D, namely enforcing epoch durations to be multiples of D (by simply requiring the epoch-based policy to wait until the next multiple of D after all packets from the start of the epoch are evacuated), which allows the Markovian nature of the system to be maintained.
   We conjecture that the epoch-based policy can be shown to be stabilizing for any general stationary and ergodic arrival process, but the necessary extension of the proof remains open at this stage.
5) A policy, which seems to be more amenable to analysis under stationary and ergodic arrivals is a *frame-based policy* which operates in periods. At each period n, beginning at time S_n, a number of packets are processed. The packets under processing in period n have all arrived in the system before S_n and correspond to a frame of arrivals of fixed duration F. In particular, during the n^{th} period, only arrivals from the frame $[I_{n-1}, I_n)$ are processed, where $I_n - I_{n-1} = F$, hence $I_n \triangleq nF$. The time to evacuate all the arrivals in the interval $[I_{n-1}, I_n)$ is random, depending on the number of arrivals as well as other random events and we denote it with $T_n(F)$. Note that if there is only one system state, then if the arrival process is stationary, $T_n(F)$ is a stationary process as well.
   Before the start of period $n+1$, a waiting time is added if $S_n + T_n(F) < I_n + F$. This waiting is imposed in order to ensure that $I_{n+1} - I_n = F$. By letting $D_n = S_n - I_n$ denote the lag process, it can be seen that
   $$D_{n+1} = (D_n + T_n(F) - F)^+ .$$
   Note that this equation is of the same form as the recursion relating the queue size in a discrete G/G/1 queue with ”arrival rate” (per slot) $T(F)$ and ”service rate” F. Note that if $T(\lambda) < 1$, then by picking F large enough, we can ensure that $\bar{T}(F) < F$ i.e., ”arrival rate” is less than the ”service rate”. We conjecture that this policy stabilizes the queue sizes under stationary and ergodic arrivals. However the policy is unattractive in practice since it induces very large delays even for small arrival rates.

VI. Application: Capacity and Stability regions of Broadcast Erasure Channel with Feedback

Consider a communication system consisting of a single transmitter and a set $\mathcal{N} \triangleq \{1, 2, \ldots, n\}$ of receivers/users (we hereafter use these two terms interchangeably). The transmitter has n infinite queues where packets destined to each of the receivers are stored. Packets consist of L bits and are transmitted within one slot. The channel is modeled as memoryless broadcast erasure (BE), so that each broadcast packet is either received unaltered by a user or is “erased” (i.e. the user does not receive the packet, but knows that a packet was sent). The latter case is equivalent to considering that the user receives the

special symbol E, which is distinct from any other possible transmitted symbol and does not map to a physical packet (since it models an erasure). To complete the description of the system we also need to specify the outputs when no packet is sent by the transmitter, i.e., the slot is empty: in this case we assume that all receivers realize that the slot is empty. An empty slot will be denoted by $\varnothing$. Equivalently, we may view "no transmission" as transmission of a special symbol $\varnothing$.

In information-theoretic terms, the broadcast erasure channel under consideration is described by the tuple $(\mathcal{X}, (\mathcal{Y}_i \in \mathcal{N}),\ p(\boldsymbol{Y}_l|X_l))$, where $\mathcal{X}$ is the input symbol alphabet, $\mathcal{Y}_i = \mathcal{Y} = \mathcal{X} \cup \{E\}$ is the output symbol alphabet for user i, and $p(\boldsymbol{Y}_l|X_l)$ is the probability of having, at slot l, output $\boldsymbol{Y}_l = (Y_{i,l}, i \in \mathcal{N})$ for a broadcast input symbol X_l. The memoryless property implies that $p(\boldsymbol{Y}_l|X_l)$ is independent of l, so that it is simply written as $p(\boldsymbol{Y}|X)$. We denote by $\epsilon_{\mathcal{N}_E}$, $\mathcal{N}_E \subseteq \mathcal{N}$, the (common) probability that a transmitted packet (i.e. a symbol in $\mathcal{X} - \{\varnothing\}$) is erased by all users in the set $\mathcal{N}_E$. To avoid unnecessary complications we assume in the following that $\epsilon_{\{i\}} < 1$ for all i. Note that for the empty slot (symbol $\varnothing$) we have $\Pr(\boldsymbol{Y} = (\varnothing, ...\varnothing)\,|X = \varnothing) = 1$.

We assume that there is feedback from the users to the transmitter, so that at the end of each slot l, all users inform the transmitter whether the symbol was received or not (essentially, a simple ACK/NACK) through an error-free zero-delay control channel.

We define two regions for this channel, the information theoretic capacity region, and the stability region. The information theoretic capacity region describes transmission rates under which it is possible to transmit sets of messages (one for each user) placed at the transmitter by using proper encoding, so that all users receive the messages destined to them with arbitrarily small probability of error. For the stability region, Definition 7 is used, under the assumption that packets arrive randomly to the system. We assume that packets are transmitted using a proper encoding, such that they are decoded by the receivers with *zero* probability of error.

We now give a precise definition of the two regions, and show in the following that they are identical.

Information theoretic capacity region

A channel code, denoted as $c_l = (M_1, \dots, M_n, l)$, for the broadcast channel with feedback is defined as the aggregate of the following components (this is an extension of the standard capacity definition of [7] to n users):

- Message sets $\mathcal{W}_i$ of size $|\mathcal{W}_i| = M_i$ for each user $i \in \mathcal{N}$, where $|\cdot|$ denotes set cardinality. Denote the message that needs to be communicated as $\boldsymbol{W} \triangleq (W_i, i \in \mathcal{N}) \in \mathcal{W}$, where $\mathcal{W} = \mathcal{W}_1 \times \ldots \times \mathcal{W}_n$. For our purposes it is helpful to interpret the message set $\mathcal{W}_i$ as follows: assume that user i needs to decode a given set $\mathcal{K}_i$ of L-bit packets. Then, $\mathcal{W}_i$ is the set of all possible $|\mathcal{K}_i|\,L$ bit sequences, so that it holds $|\mathcal{W}_i| = M_i = 2^{|\mathcal{K}_i|L}$. Henceforth we will assume this relation.
- An encoder that transmits, at slot t, a symbol $X_t = f_t(\boldsymbol{W}, \hat{\boldsymbol{Y}}^{t-1})$, based on the value of $\boldsymbol{W}$ and all previously gathered feedback $\hat{\boldsymbol{Y}}^{t-1} \triangleq (\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_{t-1}), \boldsymbol{Y}_k = (Y_{1,k}, ..., Y_{n,k})$. X_1 is a function of $\boldsymbol{W}$ only. A total of l symbols are transmitted for message $\boldsymbol{W}$.
- n decoders, one for each user $i \in \mathcal{N}$, represented by the decoding functions $g_i : \mathcal{Y}^l \to \mathcal{W}_i$ that map Y_i^l, where $Y_i^l \triangleq (Y_{i,1}, \dots, Y_{i,l})$ is the sequence of symbols received by user i during the l slots, to a message in $\mathcal{W}_i$.

In the following we write $(M_1, \dots, M_n, l)$ to denote the code c_l, with the understanding that the full specification requires all the components described above. The probability of erroneous decoding is defined as $q_l^e = \Pr(\cup_{i \in \mathcal{N}} \{g_i(Y_i^l) \neq W_i\})$, where it is assumed that the messages are selected according to the uniform distribution from $\mathcal{W}$. The rate $\boldsymbol{R}$ for this code, measured in information bits per transmitted symbol, is now defined as the vector $\boldsymbol{R} = (R_i : i \in \mathcal{N})$ with $R_i = (\log_2 M_i)/l$. Hence, it holds $R_i = |\mathcal{K}_i|\,L/l = r_i L$, where $r_i = |\mathcal{K}_i|\,/l$ is the rate of the code in packets per slot, and the bits of each packet are uniformly distributed and independent of the bits of the other packets. For our purposes, it will be convenient to define the capacity region of the system in terms of the rate vector $\boldsymbol{r} = \boldsymbol{R}/L$.

A vector rate $\boldsymbol{r} = (r_1, \dots, r_n)$ is achievable if there exists a sequence $\{c_l\}_{l=1}^{\infty}$ of codes $(2^{\lceil l r_1 \rceil L}, \dots, 2^{\lceil l r_n \rceil L}, l)$ such that $q_l^e \to 0$ as $l \to \infty$. The capacity region $\mathcal{C}$ of the system is the closure of the set of achievable rates.

Stochastic Arrivals: Definitions of admissible policies

As in Section II, we assume that packets arrive randomly to the system according to the stochastic process $\boldsymbol{A}(t)$ and are stored in infinite buffers at the transmitter. We denote by $\boldsymbol{\mathcal{A}}(t)$ the content of these messages, i.e., $\boldsymbol{\mathcal{A}}(t) = (\boldsymbol{\mathcal{A}}_1(t), ..., \boldsymbol{\mathcal{A}}_n(t))$ where $\boldsymbol{\mathcal{A}}_i(t) = \left(p_{i,1}(t), ..., p_{i,A_i(t)}(t)\right)$, and $p_{i,j}(t)$ denotes the sequences of bits corresponding to the jth packet with destination node i that arrived at the transmitter at time t — if no packets arrive we consider that $\boldsymbol{\mathcal{A}}_i(t)$ is the empty set. We assume that $p_{i,j}(t)$ are uniformly distributed and mutually independent. We denote $\hat{\boldsymbol{\mathcal{A}}}^t \triangleq (\boldsymbol{\mathcal{A}}(0), \dots, \boldsymbol{\mathcal{A}}(t))$ to be the contents of all packet arrivals up to time t.

An admissible policy consists of

- An encoder that transmits, at slot t, a symbol $X_t = f_t(\hat{\boldsymbol{Y}}^{t-1}, \hat{\boldsymbol{\mathcal{A}}}^t)$, based on all previously gathered feedback, $\hat{\boldsymbol{Y}}^{t-1} = (\boldsymbol{Y}_1, \dots, \boldsymbol{Y}_{t-1})$, and the contents of packet arrivals up to time t, $\hat{\boldsymbol{\mathcal{A}}}^t$.

- n decoders, one for each user $i \in \mathcal{N}$, represented by the decoding set-valued functions $g_{i,t}\left(Y_i^t\right)$ that at time t maps Y_i^t to a subset of the packets that have arrived up to time $t-1$ with destination node i, i.e.,

$$\mathcal{D}_t \subseteq \left\{p_{i,j}\left(\tau\right):\ \tau \leq t-1,\ 1 \leq j \leq A_i\left(t\right)\right\}.$$

A packet is decoded the first time it is included in $\mathcal{D}_t$. We set the requirement that packet decoding is correct with probability one. Note that there is at least one policy that satisfies this requirement: this is the time-sharing policy where packets destined to destination i are (re)transmitted, in First-Come-First-Served order, until successful reception in slots specifically assigned to i, say slots $jn+i-1\,, j=0,1,....$ We call such a policy One By One (OBO) policy, π_O. For this policy it can be easily seen that

$$\bar{T}_{\pi_O}\left(\boldsymbol{k}\right) \leq \sum_{i=1}^{n} C_i k_i + C_0,$$

where C_i depends on the erasure probabilities, but not on $\boldsymbol{k}$. Hence, π_O satisfies (5).

In order to apply the stability definition 7 to the class of policies specified above, we must define the time instant at which a packet leaves the system. There are two ways to define this instant. According to the first, a packet is considered to leave the system when it is correctly decoded by the destination receiver. While this definition make sense if one is interested in packet delivery times, it does not capture the fact that a decoded packet may still be needed for further encoding and decoding, in which case the packet will keep occupying buffer space even after its correct decoding. Also, the feedback information may need to be stored in the buffers of the transmitter if needed for further encoding. To capture buffer requirements we assume that each of the receivers has infinite buffers where received packets are stored. We next introduce a second definition of queue size, where we take into account the following.

1) Each transmission results in storing at most n packets, one at each receiver. These packets may be functions of “native” packets that have arrived exogenously at the transmitter, as well as the feedback received at the transmitter. Hence, in this case packets may be generated internally to the system during its operation.
2) A packet stored at a receiver buffer departs when it is not needed for further decoding.
3) A feedback packet is stored at the transmitter until it is not used for further encoding.
4) A native packet departs from the receiver if a) it has been decoded by the receiver to which it is destined and b) it is not used for further encoding.

If $Q_D^{\pi}\left(t\right)$ and $Q_B^{\pi}\left(t\right)$ respectively are the sum of queue sizes under π according to the previous two definitions of packet departure time (Delay, Buffer), it holds, $Q_D^{\pi}\left(t\right) \leq Q_B^{\pi}\left(t\right)$. Hence, if $\mathcal{S}_D$ and $\mathcal{S}_B$ are respectively the stability regions according to the two definitions, it holds

$$\mathcal{S}_B \subseteq \mathcal{S}_D. \tag{24}$$

Relation between Capacity and Stability Regions

The distributed nature of the channel introduces some new issues that must be addressed in order to apply the results of the previous sections. Specifically, while the transmitter has full knowledge of the system through the channel feedback, this is not the case for the receivers. Transferring appropriate information to the receivers takes extra slots which must be accounted for.

Note first that there are some differences in the information available at the receivers in the definition of the two regions given above. Specifically, in the capacity region definition, it is assumed that the receivers know the number of packets at the transmitter when the algorithm starts. On the other hand, when arrivals are stochastic, this information cannot be assumed a priori and if needed it must be communicated to the receivers. Also, in the capacity definition, all receivers under any admissible coding know implicitly the instant t at which the decoding process stops. For the stochastic arrival model, however, under a general evacuation policy, this may not be the case. Note that the One-By-One policy π_O does not need the information regarding the number of packets at the transmitter when the system starts. Also, an evacuation policy that is based on π_O can be easily modified to inform the receivers about the end of the decoding process: when all packets to destination i are transmitted, an empty slot is transmitted in the next slot allocated to i, informing all receivers of this event. Hence if the last packet is delivered to the appropriate destination at time t, all receivers will know at time $t+1$ that all packets are evacuated. Note that (5) still holds under this modification. We denote this modified policy as π_O^e.

Since it can be preagreed which evacuation policy to employ when a given number of packets $\boldsymbol{k}$ is initially at the transmitter, once that number is known by all receivers, the employed evacuation policy is also known by the receivers.

In the following, we initially assume the following conditions (these conditions will be removed later).

- When an evacuation policy starts, the number of packets at the transmitter is known to the receivers.
- An evacuation policy ensures that all receivers realize the end of the evacuation process at some time t, which is defined as the end of the evacuation process.

Under these conditions, the arguments of Lemma 5 apply and hence $\bar{T}^*(\boldsymbol{k})$ is again subadditive (we omit the subscript describing states since the system under discussion has just a single state). Also, the arguments for (7) still hold (note that by

placing the "dummy" packet in the argument last in the transmitter queue corresponding to receiver i, this receiver knows that this packet contains no information and hence decoding error does not occur). Hence Theorem 6 holds for the current model.

We now claim that under the above stated assumptions,

Lemma 16. *It holds,*

$$\mathcal{C} = \mathcal{R} \triangleq \left\{ \boldsymbol{r} \geq \boldsymbol{0} : \hat{T}\left(\boldsymbol{r}\right) \leq 1 \right\}.$$

Proof: We first show that $\mathcal{R} \subseteq \mathcal{C}$. For this, it suffices to show that if for some $\boldsymbol{r}$ it holds $\hat{T}\left(\boldsymbol{r}\right) < 1$, then there is a sequence of codes $c_l = (2^{\lceil l r_1 \rceil L}, \ldots, 2^{\lceil l r_n \rceil L}, l)$ with $q_l^e \to_{l\to\infty} 0$. Select $\delta > 0$ such that

$$\hat{T}\left(\boldsymbol{r}\right) + 3\delta < 1. \tag{25}$$

In the following, we denote, for any positive integers l and l_0, $\alpha_l = \left\lfloor \frac{l}{l_0} \right\rfloor$ and $\beta_l = l \mod l_0$, i.e.

$$l = \alpha_l l_0 + \beta_l, \; 0 \leq \beta_l < l_0$$

It follows that

$$\lceil l r_i \rceil \leq \left(\alpha_l + 1\right) \lceil l_0 r_i \rceil. \tag{26}$$

Select and fix l_0 large enough so that

$$\frac{\bar{T}^*\left(\lceil l_0 \boldsymbol{r} \rceil\right)}{l_0} \leq \hat{T}\left(\boldsymbol{r}\right) + \delta. \tag{27}$$

Select an evacuation policy π_{l_0} such that

$$\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r} \rceil\right) \leq \bar{T}^*\left(\lceil l_0 \boldsymbol{r} \rceil\right) + l_0 \delta, \tag{28}$$

Consider the following sequence of codes c_l for transmitting $\lceil l\boldsymbol{r} \rceil$ packets.

a) Use π_{l_0} to transmit successively $\alpha_l + 1$ batches of $\lceil l_0 \boldsymbol{r} \rceil$ packets (the last batch may contain dummy packets) until they are decoded by all receivers. Let $T^j_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r} \rceil\right)$ be the (random) time it takes to transmit the j-th batch, and

$$\tilde{T}^l_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r} \rceil\right) = \sum_{j=1}^{\alpha_l+1} T^j_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r} \rceil\right)$$

b) If

$$\tilde{T}^l_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r} \rceil\right) \leq l$$

all packets are correctly decoded; else declare an error.

The probability of error for the sequence c_l is computed as follows. Observing that

$$\lim_{l\to\infty} \alpha_l = \infty, \; \lim_{l\to\infty} \frac{\beta_l}{\alpha_l} = 0,$$

and taking into account (25), pick $\tilde{l}$ large enough so that for all $l \geq \tilde{l}$ it holds,

$$\begin{aligned}
\frac{\alpha_l}{\alpha_l + 1}\left(1 + \frac{\beta_l}{l_0 \alpha_l}\right) &= \left(1 - \frac{1}{\alpha_l + 1}\right)\left(1 + \frac{\beta_l}{l_0 \alpha_l}\right) \\
&= 1 + \frac{\beta_l}{l_0 \alpha_l} - \frac{1}{\alpha_l + 1}\left(1 + \frac{\beta_l}{l_0 \alpha_l}\right) \\
&\geq 1 - \frac{1}{\alpha_l + 1}\left(1 + \frac{\beta_l}{l_0 \alpha_l}\right) \\
&\geq \hat{T}\left(\boldsymbol{r}\right) + 3\delta
\end{aligned} \tag{29}$$

Then,

$$
\begin{aligned}
q_l^e &= \Pr\left\{\tilde{T}^l_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right) > l\right\}\\
&= \Pr\left\{\sum_{j=1}^{\alpha_l+1} T^j_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right) > \alpha_l l_0 + \beta_l\right\}\\
&= \Pr\left\{\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} > \frac{\alpha_l}{\alpha_l+1}\left(1+\frac{\beta_l}{l_0\alpha_l}\right)\right\}\\
&\le \Pr\left\{\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} > \hat{T}\left(\boldsymbol{r}\right) + 3\delta\right\} \qquad \text{by} \quad (29)\\
&\le \Pr\left\{\left|\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} - \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0}\right| > \hat{T}\left(\boldsymbol{r}\right) - \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0} + 3\delta\right\}\\
&\le \Pr\left\{\left|\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} - \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0}\right| > \hat{T}\left(\boldsymbol{r}\right) - \frac{\bar{T}^*\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0} + 2\delta\right\} \qquad \text{by } (28)\\
&\le \Pr\left\{\left|\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} - \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0}\right| > \delta\right\} \qquad \text{by } (27).
\end{aligned}
$$

Due to the memorylessness of the channel and the fact that the bits in the packet contents are i.i.d, the random variables $T^j_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right),\ j=1,2...$ are i.i.d. Using the fact that $\alpha_l \rightarrow_{l\rightarrow\infty} \infty$, we conclude

$$
\lim_{l\rightarrow\infty}\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} = \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0}
$$

which implies that

$$
\lim_{l\rightarrow\infty} q_l^e = \lim_{l\rightarrow\infty}\Pr\left\{\left|\frac{\sum_{j=1}^{\alpha_l+1}\frac{T^j_{\pi_{l_0}}(\lceil l_0 \boldsymbol{r}\rceil)}{l_0}}{\alpha_l+1} - \frac{\bar{T}_{\pi_{l_0}}\left(\lceil l_0 \boldsymbol{r}\rceil\right)}{l_0}\right| > \delta\right\} = 0.
$$

Next we show that $\mathcal{C} \subseteq \mathcal{R}$. Assume that $\boldsymbol{r} \in \mathcal{C}$ so that there is a sequence of coding algorithms c_l with rate $\boldsymbol{r}$ whose error probability approaches zero in the limit as $l \rightarrow \infty$. We then construct an evacuation policy π_l for evacuating $\lceil l\boldsymbol{r}\rceil$ packets as follows.

a) For $\epsilon > 0$, select l so that $q_l^e < \epsilon$.

b) Follow the steps of c_l for the first l slots.

c) If all receivers decoded correctly, leave slot $l+1$ empty, thus signaling to all receivers the end of the decoding process.

e) Else (i.e., if any of the receivers makes an error), send a dummy packet in slot $l+1$ (thus informing the receivers that decoding continues) and resend all the $\lceil l\boldsymbol{r}\rceil$ packets using the one-by-one policy π_O.

Note that, since the transmitter knows g_i and, through the received feedback, the sequence received by i, it knows whether a receiver makes an error and hence the third step above is implementable.

We compute the average evacuation time of π_l as follows. Let $\mathcal{E}$ be the event that all destinations have decoded the packets in l slots. Then, since on $\mathcal{E}^c$ it holds

$$
T_{\pi_l}\left(\lceil l\boldsymbol{r}\rceil\right) = l + T_{\pi_O}\left(\lceil l\boldsymbol{r}\rceil\right) + 1,
$$

$T_{\pi_O}\left(\lceil l\boldsymbol{r}\rceil\right)$ is independent of $\mathcal{E}^c$, and by choice $\Pr\left\{\mathcal{E}^c\right\} = q_l^e < \epsilon$, therefore we have

$$
\begin{aligned}
\mathbb{E}\left[T_{\pi_l}\left(\lceil l\boldsymbol{r}\rceil\right) 1_{\mathcal{E}^c}\right] &= l\Pr\left\{\mathcal{E}^c\right\} + \Pr\left\{\mathcal{E}^c\right\}\left(\bar{T}_{\pi_O}+1\right)\\
&\le l\epsilon + \epsilon\left(C_1\sum_{i=1}^n \lceil l r_i\rceil + C_0 + 1\right).
\end{aligned}
$$

Taking into account that $T_{\pi_l}(\lceil l\boldsymbol{r}\rceil) = l+1$ on $\mathcal{E}$,

$$\begin{aligned}\mathbb{E}\left[T_{\pi_l}(\lceil l\boldsymbol{r}\rceil)\right] &= \mathbb{E}\left[T_{\pi_l}(\lceil l\boldsymbol{r}\rceil)\,1_{\mathcal{E}}\right] + \mathbb{E}\left[T_{\pi_l}(\lceil l\boldsymbol{r}\rceil)\,1_{\mathcal{E}^c}\right] \\ &\le l+1+\epsilon\left(l + C_1\sum_{i=1}^{n}\lceil lr_i\rceil + C_0 + 1\right)\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\bar{T}_A^*(\lceil l\boldsymbol{r}\rceil)}{l} &\le \frac{\bar{T}_{\pi_l}(\lceil l\boldsymbol{r}\rceil)}{l} \\ &\le 1 + \frac{1}{l} + \epsilon\cdot\frac{l + C_1\sum_{i=1}^{n}\lceil lr_i\rceil + C_0 + 1}{l}\end{aligned}$$

Considering the limit as $l\to\infty$, we obtain,

$$\hat{T}_A(\boldsymbol{r}) \le 1 + \epsilon\left(C_1\sum_{i=1}^{n} r_i + 1\right)$$

and since ϵ is arbitrary we conclude

$$\hat{T}_A(\boldsymbol{r}) \le 1.$$

■

It remains to relate $\mathcal{R}$ to $\mathcal{S}_D$ and $\mathcal{S}_B$ under the current model. Revisit the proof of Theorem 9, and use a policy $\pi_0 \in \mathcal{S}_D$ for the first l slots. It all packets are decoded correctly by slot l, leave slot $l+1$ empty, thus informing all receivers of successful decoding. Else send a dummy packet in slot $l+1$ and afterwards apply the One-By-One policy π_O as policy π_h in the proof to evacuate the remaining packets. With these modifications, the proof can be used to show that

$$\mathcal{S}_D \subseteq \mathcal{R}. \tag{30}$$

We now consider the implementation of the Epoch Based policy π_ϵ under the current model. This policy selects a particular evacuation policy for each epoch, which is a function of the number of packets $\boldsymbol{k}$ at the beginning of the epoch. In order to implement π_ϵ in the current model, the receivers must generally know $\boldsymbol{k}$ at the beginning of an epoch. The transfer of information about the number $\boldsymbol{k}$ is done by transmitting $O\left(\sum_{i=1}^{n}\log(k_i+1)\right)$ packets (for example, using the One-by-One policy π_O) and hence the average number of slots to achieve this transfer is $O\left(\sum_{i=1}^{n}\log(k_i+1)\right)$. This increases the length of the evacuation period but since the increase is logarithmic in the number of packets, it does not affect the stability arguments. Note also that once an epoch ends, all $\boldsymbol{k}$ packets, as well as the feedback information and the packets stored at the receivers can be discarded since they are not used for further decoding by π_ϵ. Hence we conclude that

$$\mathcal{R} \subseteq \mathcal{S}_B \tag{31}$$

Taking into account (24), (30), (31) we finally conclude,

Theorem 17. *It holds,*

$$\mathcal{C} = \mathcal{R} = \mathcal{S}_B = \mathcal{S}_D.$$

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Appendix A
Proof of Theorem 6

An analogous to Theorem 6 has been derived in [8] for subadditive functions defined on $\mathbb{R}^n$. The extension of Critical Evacuation Time Function to $\mathbb{R}_0^n$ given in (10) is not necessarily subadditive and hence we need different arguments to show the result, albeit using similar ideas.

Let $f(\boldsymbol{k}) : \mathbb{N}_0^n \to \mathbb{R}_0$ be a subadditive function. Let $\mathcal{U}$ be the set of n-dimensional vectors whose coordinates are either zero or one, and define,

$$U = \max_{\boldsymbol{u} \in \mathcal{U}} f(\boldsymbol{u}).$$

We will need the following lemma.

Lemma 18. *For any $\boldsymbol{k} \in \mathbb{N}_0^n - \{\boldsymbol{0}\}$, it holds*

$$f(\boldsymbol{k}) \leq U \max_i k_i.$$

Proof: Assume without loss of generality that for some $c \leq n$, $0 < k_1 \leq k_2 \leq ... \leq k_c$ and, in case $c < n$, then $k_{c+1} = \ldots = k_n = 0$. Write,

$$\boldsymbol{k} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \sum_{i=1}^{c} (k_i - k_{i-1}) \boldsymbol{u}_i,$$

where $k_0 = 0$ and

$$u_{i,j} = \begin{cases} 0 & \text{if } \ i > 1 \quad \text{and} \quad j = 1, ..., i-1, \\ 1 & \text{if } \ j = i, ..., c \\ 0 & \text{if } \ j > c \end{cases}$$

By subadditivity we have,

$$\begin{aligned} f(\boldsymbol{k}) &\leq \sum_{i=1}^{c} (k_i - k_{i-1}) f(\boldsymbol{u}_i) \\ &\leq U k_c \end{aligned}$$

■

Next we extend the definition of $f(\boldsymbol{k})$ to $\mathbb{R}_0^n$ by defining

$$f(\boldsymbol{r}) = f(\lceil \boldsymbol{r} \rceil), \ \boldsymbol{r} \in \mathbb{R}_0^n.$$

We then have the following theorem.

Theorem 19. *For any $\boldsymbol{r} \in \mathbb{R}_0^n$, the limit function*

$$\hat{f}(\boldsymbol{r}) = \lim_{t \to \infty} \frac{f(t\boldsymbol{r})}{t} \tag{32}$$

exists, is finite and positively homogeneous.

Proof: Assume without loss of generality that $r_1 \geq r_2 \geq ... \geq r_n$. If $r_1 = 0$ then $\boldsymbol{r} = \boldsymbol{0}$ and (32) is obvious. Assume next that for some c, $1 \leq c \leq n$, $r_c > 0$ and $r_{c+1} = 0$. For consistency define $r_{n+1} = 0$.

Let $\epsilon > 0$ and $\beta = \liminf_{t \to \infty} f(t\boldsymbol{r})/t \geq 0$. Using Lemma 18 we have,

$$\begin{aligned} \frac{f(t\boldsymbol{r})}{t} &= \frac{f(\lceil t\boldsymbol{r} \rceil)}{t} \\ &\leq U \frac{\max_i \{\lceil t r_i \rceil\}}{t} \\ &< U \frac{\max_i \{t r_i\} + 1}{t} \\ &= U \left(\max_i \{r_i\} + \frac{1}{t} \right) \end{aligned}$$

Hence, $\beta < \infty$.

To show existence of the limit in (32), it suffices to show that

$$\limsup_{t \to \infty} \frac{f(t\boldsymbol{r})}{t} \leq \beta + \delta(\epsilon), \tag{33}$$

where $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$.

By definition of β, there are infinitely many t, such that $f(t\boldsymbol{r})/t \leq \beta+\epsilon$. Since we also have

$$r_i \leq \frac{\lceil tr_i \rceil}{t} < r_i + \frac{1}{t}, \tag{34}$$

we can pick t_0 large enough so that the following inequalities hold.

$$\frac{f(t_0\boldsymbol{r})}{t_0} \leq \beta+\epsilon, \tag{35}$$

$$r_i \leq \frac{\lceil t_0 r_i \rceil}{t_0} < r_i + \epsilon, \;\; i=1,...,k. \tag{36}$$

Using Euclidean division, write for $i=1,...,c$

$$\lceil tr_i \rceil = l_{t,i}\lceil t_0 r_i \rceil + \upsilon_{t,i},\; 0 \leq \upsilon_{t,i} \leq \lceil t_0 r_i \rceil - 1 \tag{37}$$

If $c<n$, define also,

$$l_{t,i} = \upsilon_{t,i} = 0,\; i=c+1,...,n \tag{38}$$

We then have,

$$\begin{aligned} f(t\boldsymbol{r}) &= f(\lceil t\boldsymbol{r} \rceil) \\ &= f(l_{t,1}\lceil t_0 r_1 \rceil + \upsilon_{t,1},...,l_{t,n}\lceil t_0 r_n \rceil + \upsilon_{t,n}) \\ &\leq f(l_{t,1}\lceil t_0 r_1 \rceil,...,l_{t,n}\lceil t_0 r_n \rceil) + f(\boldsymbol{\upsilon}_t) \;\; \text{by subadditivity} \end{aligned} \tag{39}$$

Next, write

$$\begin{bmatrix} l_{t,1}\lceil t_0 r_1 \rceil \\ \vdots \\ l_{t,n}\lceil t_0 r_n \rceil \end{bmatrix} = \sum_{j=1}^{c} (l_{t,j} - l_{t,j-1})\, \boldsymbol{v}_j,$$

where $l_{t,0}=0$ and the ith coordinate of $\boldsymbol{v}_j$, $v_{j,i}$, is defined for $1 \leq j \leq c$ as,

$$v_{j,i} = \begin{cases} 0 & \text{if } \; j \neq 1 \quad \text{and} \quad i=1,...j-1, \\ \lceil t_0 r_i \rceil & \text{if } \; i=j,...,n \end{cases} \tag{40}$$

Notice that since $r_j \geq r_{j+1}$, it holds, $l_{t,j-1} \leq l_{t,j}$, $1 \leq j \leq c$. Using subadditivity, we then have from (39),

$$f(t\boldsymbol{r}) \leq \sum_{j=1}^{c} (l_{t,j} - l_{t,j-1})\, f(\boldsymbol{v}_j) + f(\boldsymbol{\upsilon}_t)$$

Hence,

$$\begin{aligned} \frac{f(t\boldsymbol{r})}{t} &\leq \sum_{j=1}^{c} \frac{(l_{t,j} - l_{t,j-1})\, t_0}{t} \frac{f(\boldsymbol{v}_j)}{t_0} + \frac{f(\boldsymbol{\upsilon}_t)}{t} \\ &= \frac{l_{t,1}t_0}{t}\frac{f(t_0\boldsymbol{r})}{t_0} + \sum_{j=2}^{c} \frac{(l_{t,j} - l_{t,j-1})\, t_0}{t} \frac{f(\boldsymbol{v}_j)}{t_0} + \frac{f(\boldsymbol{\upsilon}_t)}{t} \end{aligned} \tag{41}$$

By (37), (38), $\boldsymbol{\upsilon}_t$ takes a finite number of values, hence $f(\boldsymbol{\upsilon}_t)$ is a bounded sequence, and

$$\lim_{t\to\infty} \frac{f(\boldsymbol{\upsilon}_t)}{t} = 0.$$

Also, from (34), (36) and (37) we have for $1 \leq i \leq c$,

$$r_i \leq \frac{\lceil tr_i \rceil}{t} = \frac{l_{t,i}t_0}{t}\frac{\lceil t_0 r_i \rceil}{t_0} + \frac{\upsilon_{t,i}}{t} < \frac{l_{t,i}t_0}{t}(r_i+\epsilon) + \frac{\upsilon_{t,i}}{t},$$

$$r_i + \frac{1}{t} > \frac{\lceil tr_i \rceil}{t} = \frac{l_{ti}t_0}{t}\frac{\lceil t_0 r_i \rceil}{t_0} + \frac{\upsilon_{ti}}{t} \geq \frac{l_{ti}t_0}{t} r_i + \frac{\upsilon_{t,i}}{t},$$

hence, using the fact that $\boldsymbol{\upsilon}_t$ is a bounded sequence, we conclude

$$1 - \frac{\epsilon}{r_c} \leq 1 - \frac{\epsilon}{r_i} \leq \frac{r_i}{r_i+\epsilon} \leq \liminf_{t\to\infty} \frac{l_{t,i}t_0}{t} \leq \limsup \frac{l_{t,i}t_0}{t} \leq 1.$$

Taking into account the latter inequalities and (35) we have from (41),

$$\begin{aligned}
\limsup_{t\to\infty}\frac{f\left(t\boldsymbol{r}\right)}{t} &\le (\beta+\epsilon)\limsup_{t\to\infty}\frac{l_{t,1}t_0}{t}+\sum_{j=2}^{c}\left(\limsup_{t\to\infty}\frac{l_{t,j}t_0}{t}-\liminf_{t\to\infty}\frac{l_{t,j-1}t_0}{t}\right)\frac{f\left(\boldsymbol{v}_j\right)}{t_0}\\
&\le \beta+\epsilon+\frac{\epsilon}{r_c}\sum_{j=2}^{c}\frac{f\left(\boldsymbol{v}_j\right)}{t_0}\\
&\le \beta+\epsilon+\frac{\epsilon}{r_c}Uc\frac{\max_i\lceil t_0 r_i\rceil}{t_0}\ \text{ by Lemma 18 and (40)}\\
&\le \beta+\epsilon+\frac{\epsilon}{r_c}Un\left(r_1+\epsilon\right)\ \text{ by (36)}
\end{aligned}$$

Hence (33) holds with $\delta(\epsilon)=\epsilon+\frac{\epsilon}{r_c}Un\left(r_1+\epsilon\right)$.

Positive homogeneity follows immediately since for $\alpha\ge 0$,

$$\hat{f}\left(\alpha\boldsymbol{r}\right)=\lim_{t\to\infty}\frac{f\left(t\alpha\boldsymbol{r}\right)}{t}=\alpha\lim_{t\to\infty}\frac{f\left(t\alpha\boldsymbol{r}\right)}{\alpha t}=\alpha\hat{f}\left(\boldsymbol{r}\right).$$

■

The next lemma is needed to establish further properties of $f\left(\boldsymbol{k}\right)$ in Theorem 21 below.

Lemma 20. *Let a subadditive function $f\left(\boldsymbol{k}\right)$, $\boldsymbol{k}\in\mathbb{N}_0^n$ satisfy*

$$f\left(\boldsymbol{k}\right)-f\left(\boldsymbol{k}+\boldsymbol{e}_i\right)\le D_0 \tag{42}$$

Then the following holds with $D=\max\left\{f\left(\boldsymbol{e}_1\right),...,f\left(\boldsymbol{e}_n\right),D_0\right\}$.

$$\left|f\left(\boldsymbol{k}\right)-f\left(\boldsymbol{k}+\boldsymbol{e}_i\right)\right|\le D,\ \text{for all } i=1,..,n. \tag{43}$$

$$\left|f\left(\boldsymbol{k}\right)-f\left(\boldsymbol{m}\right)\right|\le D\sum_{i=1}^{n}\left|k_i-m_i\right| \tag{44}$$

$$\left|f\left(\boldsymbol{r}\right)-f\left(\boldsymbol{s}\right)\right|< D\sum_{i=1}^{n}\left|r_i-s_i\right|+nD \tag{45}$$

Proof: By subadditivity,

$$f\left(\boldsymbol{k}+\boldsymbol{e}_i\right)\le f\left(\boldsymbol{k}\right)+f\left(\boldsymbol{e}_i\right)$$

hence,

$$f\left(\boldsymbol{k}+\boldsymbol{e}_i\right)-f\left(\boldsymbol{k}\right)\le\max_i\bar{T}^*\left(\boldsymbol{e}_i\right)\doteq D_1$$

Taking into account (42) we conclude,

$$\left|f\left(\boldsymbol{k}+\boldsymbol{e}_i\right)-f\left(\boldsymbol{k}\right)\right|\le\max\left\{D_1,D_0\right\}\doteq D$$

which shows (43).

To show (44) we use backward induction on the number c of coordinates of $\boldsymbol{k}$, $\boldsymbol{m}$ that are equal. If $c=n$ then clearly (44) holds. Let (44) hold for $c\le n$ and assume without loss of generality that $k_i=m_i$, $i=1,...,c-1$ and $k_i\ne m_i$, $i\ge c$, $k_c>m_c$. We then have

$$\begin{aligned}
\left|f\left(\boldsymbol{k}\right)-f\left(\boldsymbol{m}\right)\right| &= \left|f\left(\boldsymbol{k}\right)-f\left(k_1,...k_{c-1},m_c,k_{c+1},...,k_n\right)+f\left(k_1,...k_{c-1},m_c,k_{c+1},...,k_n\right)-f\left(\boldsymbol{m}\right)\right|\\
&\le \left|f\left(\boldsymbol{k}\right)-f\left(k_1,...k_{c-1},m_c,k_{c+1},...,k_n\right)\right|+\left|f\left(m_1,...m_{c-1},m_c,k_{c+1},...,k_n\right)-f\left(\boldsymbol{m}\right)\right|\\
&\le \left|f\left(\boldsymbol{k}\right)-f\left(k_1,...k_{c-1},m_c,k_{c+1},...,k_n\right)\right|+D\sum_{i=c+1}^{n}\left|k_i-m_i\right|\ \text{ by the inductive hypothesis.}
\end{aligned}$$

Now, write

$$\begin{aligned}
\left|f\left(\boldsymbol{k}\right)-f\left(k_1,...k_{c-1},m_c,k_{c+1},...,k_n\right)\right| &= \left|\sum_{i=0}^{k_c-m_c-1}f\left(k_1,...k_{c-1},m_c+i+1,k_{c+1},...,k_n\right)-f\left(k_1,...k_{c-1},m_c+i,k_{c+1},...,k_n\right)\right|\\
&\le \sum_{i=0}^{k_c-m_c-1}\left|f\left(k_1,...k_{c-1},m_c+i+1,k_{c+1},...,k_n\right)-f\left(k_1,...k_{c-1},m_c+i,k_{c+1},...,k_n\right)\right|\\
&\le \sum_{i=0}^{k_c-m_c-1}D\ \text{ by (43)}\\
&= D\left|k_c-m_c\right|
\end{aligned}$$

and hence,

$$\left|f\left(\boldsymbol{k}\right)-f\left(\boldsymbol{m}\right)\right|\leq D\sum_{i=c}^{n}\left|k_i-m_i\right|=D\sum_{i=1}^{n}\left|k_i-m_i\right|\ \text{ since } k_i=m_i,\ i=1,...,c-1$$

i.e., the inductive hypothesis holds for $c-1$ as well.

Finally, for (45), write

$$\begin{aligned}\left|f\left(\boldsymbol{r}\right)-f\left(\boldsymbol{s}\right)\right| &= \left|f\left(\lceil\boldsymbol{r}\rceil\right)-f\left(\lceil\boldsymbol{s}\rceil\right)\right| \\ &\leq D\sum_{i=1}^{n}\left|\lceil r_i\rceil-\lceil s_i\rceil\right| \ \text{ by (44)} \\ &< D\sum_{i=1}^{n}\left|r_i-s_i\right|+Dn,\ \text{since } \left|\lceil r_i\rceil-\lceil s_i\rceil\right|<\left|r_i-s_i\right|+1\end{aligned}$$

■

The next theorem provides further useful properties of $\hat{f}\left(\boldsymbol{r}\right)$ under condition (42).

Theorem 21. *If a subadditive function $f\left(\boldsymbol{k}\right),\ \boldsymbol{k}\in\mathbb{N}_0^n$ satisfies (42), then the limit function*

$$\hat{f}\left(\boldsymbol{r}\right)=\lim_{t\to\infty}\frac{f\left(t\boldsymbol{r}\right)}{t}$$

is subadditive, convex, Lipschitz continuous, i.e., it holds

$$\left|\hat{f}\left(\boldsymbol{r}\right)-\hat{f}\left(\boldsymbol{s}\right)\right|\leq D\sum_{i=1}^{n}\left|r_i-s_i\right|.$$

and for any sequence $\boldsymbol{r}_t\in\mathbb{R}_0^n$ such that

$$\lim_{t\to\infty}\boldsymbol{r}_t=\boldsymbol{\lambda}<\boldsymbol{\infty},$$

it holds

$$\lim_{t\to\infty}\frac{f\left(t\boldsymbol{r}_t\right)}{t}=\hat{f}\left(\boldsymbol{\lambda}\right). \tag{46}$$

Proof: To show subadditivity, we proceed as follows. Since for any $a,\ b$ it holds

$$\lceil a+b\rceil+x=\lceil a\rceil+\lceil b\rceil\ \text{ for some } x=0,1,2,$$

we write

$$\lceil t\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\rceil+\boldsymbol{x}=\lceil t\boldsymbol{r}_1\rceil+\lceil t\boldsymbol{r}_2\rceil.$$

Also, by (44)

$$\begin{aligned}f\left(\lceil t\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\rceil\right)-f\left(\lceil t\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\rceil+\boldsymbol{x}\right) &\leq D\sum_{i=}^{n}x_i \\ &\leq 2nD\end{aligned}$$

Hence,

$$\begin{aligned}f\left(t\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\right)-2nD &\leq f\left(\lceil t\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\rceil+\boldsymbol{x}\right) \\ &= f\left(\lceil t\boldsymbol{r}_1\rceil+\lceil t\boldsymbol{r}_2\rceil\right) \\ &\leq f\left(t\boldsymbol{r}_1\right)+f\left(t\boldsymbol{r}_2\right)\end{aligned}$$

Dividing the last inequality by t and taking limits shows that $\hat{f}\left(\boldsymbol{r}_1+\boldsymbol{r}_2\right)\leq\hat{f}\left(\boldsymbol{r}_1\right)+\hat{f}\left(\boldsymbol{r}_2\right)$.

Convexity follows easily from positive homogeneity and subadditivity,

$$\begin{aligned}\hat{f}\left(p\boldsymbol{r}_1+(1-p)\boldsymbol{r}_2\right) &\leq \hat{f}\left(p\boldsymbol{r}_1\right)+\hat{f}\left((1-p)\boldsymbol{r}_2\right) \\ &= p\hat{f}\left(\boldsymbol{r}_1\right)+(1-p)\hat{f}\left(\boldsymbol{r}_2\right).\end{aligned}$$

Lipschitz continuity follows easily as well from (45) by replacing $\boldsymbol{r},\ \boldsymbol{s}$ with $t\boldsymbol{r},\ t\boldsymbol{s}$, dividing by t and taking limits.

Finally let

$$\lim_{t\to\infty}\boldsymbol{r}_t=\boldsymbol{\lambda}<\boldsymbol{\infty}$$

Using (45) write

$$\begin{aligned}\left|\frac{f\left(t\boldsymbol{r}_t\right)}{t}-\hat{f}\left(\boldsymbol{\lambda}\right)\right| &= \left|\frac{f\left(t\boldsymbol{r}_t\right)}{t}-\frac{f\left(t\boldsymbol{\lambda}\right)}{t}+\frac{f\left(t\boldsymbol{\lambda}\right)}{t}-\hat{f}\left(\boldsymbol{\lambda}\right)\right| \\ &\le \left|\frac{f\left(t\boldsymbol{r}_t\right)-f\left(t\boldsymbol{\lambda}\right)}{t}\right|+\left|\frac{f\left(t\boldsymbol{\lambda}\right)}{t}-\hat{f}\left(\boldsymbol{\lambda}\right)\right| \\ &\le D\sum_{i=1}^{n}\left|r_{t,i}-\lambda_i\right|+\frac{nD}{t}+\left|\frac{f\left(t\boldsymbol{\lambda}\right)}{t}-\hat{f}\left(\boldsymbol{\lambda}\right)\right|\end{aligned}$$

Taking limits in the last inequality shows (46). ■

Theorem 6 will follow directly from Theorems 19, 21 if we verify that the critical evacuation time function satisfies (42). But this follows easily from (6) since

$$\begin{aligned}\bar{T}^*\left(\boldsymbol{k}\right)-\bar{T}^*\left(\boldsymbol{k}+\boldsymbol{e}_i\right) &= \max_s \bar{T}_s^*\left(\boldsymbol{k}\right)-\max_i \bar{T}_s^*\left(\boldsymbol{k}+\boldsymbol{e}_i\right) \\ &\le \max_s\left\{\bar{T}_s^*\left(\boldsymbol{k}\right)-\bar{T}_s^*\left(\boldsymbol{k}+\boldsymbol{e}_i\right)\right\} \\ &\le D_0, \text{ by (6)}.\end{aligned}$$

APPENDIX B
PROOF OF THEOREM 15

In the discussion that follows we use the terminology and related results in [9]. For $x, y \in \mathcal{G}$, if x *leads* to y, we write $x \rightsquigarrow y$ and if x *communicates* with y, $x \leftrightsquigarrow y$. A Markov Chain with countable state space $\mathcal{G}$ is called irreducible if all states in $\mathcal{G}$ belong to the same *essential* class, i.e., all states communicate with each other.

The proof of stability of the Epoch Based Policy is based on the following theorem, see [10], [11].

Theorem 22. *Let $\{X_m\}_{m=1}^{\infty}$ be a homogeneous, irreducible and aperiodic Markov Chain with countable state space $\mathcal{G}$. Let $v(x)$ be a nonnegative real function defined on the state space (Lyapunov function). If there exists a finite set $\mathcal{A} \subseteq \mathcal{G}$ such that $v(x) \ge \epsilon > 0$, $x \in \mathcal{A}^c = \mathcal{G} - \mathcal{A}$,*

$$\mathbb{E}\left[v(X_2)\,|X_1 = x\right] < \infty,\ x \in \mathcal{A}, \tag{47}$$

and for some δ, $1 \ge \delta > 0$,

$$\mathbb{E}\left[v(X_2)\,|X_1 = x\right] \le (1-\delta)\,v(x),\ x \in \mathcal{A}^c, \tag{48}$$

then the Markov Chain is geometrically ergodic (positive recurrent) and $\mathbb{E}\left[v\left(\widehat{X}\right)\right] < \infty$, where $\widehat{X}$ has the steady-state distribution of $\{X_m\}_{m=1}^{\infty}$.

For the general model under consideration in the current work, irreducibility and aperiodicity may not hold. Hence, we need some preparatory work to use Theorem 22. The following lemma will be useful.

Lemma 23. *Let $\{X_m\}_{m=1}^{\infty}$ be a homogeneous Markov Chain, not necessarily irreducible and/or aperiodic.*

a) With the notation of Theorem 22, conditions (47) and (48) imply

$$\mathbb{E}\left[v(X_2)\,|X_1 = x\right] \le U + (1-\delta)\,v(x) \text{ for all } x \in \mathcal{G}, \tag{49}$$

where $U = \max_{x\in\mathcal{A}} \mathbb{E}\left[v(X_2)\,|X_1 = x\right]$.

b) Conversely, if $v\left(x\right) \ge 0$ and there are constants $U > 0$, δ, δ_1, $0 < \delta_1 < \delta \le 1$, and a finite set $\mathcal{B}$ such that (49) holds and

$$\frac{U}{v\left(x\right)} \le \delta_1 \text{ for all } x \in \mathcal{B}^c, \tag{50}$$

then (47) and (48) hold with $\mathcal{A} \leftarrow \mathcal{B}$ and $\delta \leftarrow \delta - \delta_1$.

c) If (49) holds, then for $m \ge 2$,

$$\mathbb{E}\left[v(X_m)\,|X_1 = x\right] \le \frac{U}{\delta} + (1-\delta)^m\,v\left(x\right). \tag{51}$$

Proof: It is clear that (47) and (48) imply (49). Assume now that (49) and (50) hold. Then clearly (47) is satisfied for all $x \in \mathcal{B}$. Also, since the following holds for $x \in \mathcal{B}^c$,

$$\begin{aligned}\mathbb{E}\left[v(X_2)\,|X_1 = x\right] &\le \left(1-\left(\delta-\frac{U}{v\left(x\right)}\right)\right)v(x) \\ &\le \left(1-\left(\delta-\delta_1\right)\right)v(x),\end{aligned}$$

it follows that (48) is satisfied for $x \in \mathcal{B}^c$ with $\delta \leftarrow \delta - \delta_1$.

To prove (51), write

$$\begin{aligned}\mathbb{E}\left[v(X_m)\left|X_1=x\right.\right] &= \mathbb{E}\left[\mathbb{E}\left[v(X_m)\left|X_{d-1},X_1=x\right.\right]\right]\\ &\leq U+(1-\delta)\,\mathbb{E}\left[v(X_{m-1})\left|X_1=x\right.\right] \text{ by Markov property and (49)}\end{aligned}$$

and hence by induction,

$$\begin{aligned}\mathbb{E}\left[v(X_m)\left|X_1=x\right.\right] &\leq U\sum_{i=0}^{m-1}(1-\delta)^i+(1-\delta)^m\,v\left(x\right)\\ &\leq \frac{U}{\delta}+(1-\delta)^m\,v\left(x\right).\end{aligned}$$

■

The next lemma states that when (21) holds, the Markov process described in section V, namely $\{(T_m,S_m)\}_{m=1}^{\infty}$ (where T_m is the duration of the m-th epoch and S_m is the system state at the end of the m-th epoch) has the drift property described in Lemma 23.

Lemma 24. *For the Markov process $\{(T_m,S_m)\}_{m=1}^{\infty}$ define $v\left((\tau,s)\right)=\tau$. If $\hat{T}\left(\boldsymbol{\lambda}\right)<1$ then there are $U>0$ and $\delta>0$ such that.*

$$\mathbb{E}\left[v\left((T_2,S_2)\right)\left|(T_1,S_1)=(\tau,s)\right.\right]\leq U+(1-\delta)\,v\left((\tau,s)\right) \text{ for all } (\tau,s)\in\mathcal{G},$$

and (50) is also satisfied.

Proof: Using the definition of v, and the fact that given $\boldsymbol{k}_1$ and S_1, T_2 is independent of T_1, write,

$$\begin{aligned}\mathbb{E}\left[v\left((T_2,S_2)\right)\left|(T_1,S_1)=(\tau,s)\right.\right] &= \mathbb{E}\left[T_2\left|T_1=\tau,\ s_1=s\right.\right]\\ &= \mathbb{E}\left[\mathbb{E}\left[T_2\left|T_1=\tau,\ s_1=s,\boldsymbol{k}_1\left(\tau\right)\right.\right]\left|(T_1,S_1)=(\tau,s)\right.\right]\\ &= \mathbb{E}\left[\bar{T}_s^{\pi_{\boldsymbol{k}_1,s}}\left(\boldsymbol{k}_1\left(\tau\right)\right)\right]\end{aligned}\tag{52}$$

We have by construction of π_ϵ,

$$\begin{aligned}\mathbb{E}\left[\bar{T}_s^{\pi_{\boldsymbol{k}_1,s}}\left(\boldsymbol{k}_1\left(\tau\right)\right)\right] &\leq \mathbb{E}\left[\bar{T}_s^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)\right]+\epsilon\\ &\leq \mathbb{E}\left[\bar{T}^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)\right]+\epsilon.\end{aligned}\tag{53}$$

Since the arrival process vectors are i.i.d, it holds with probability 1,

$$\lim_{\tau\to\infty}\frac{\boldsymbol{k}_1\left(\tau\right)}{\tau}=\boldsymbol{\lambda},$$

and,

$$\begin{aligned}\lim_{\tau\to\infty}\frac{\bar{T}^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)}{\tau} &= \lim_{\tau\to\infty}\frac{\bar{T}^{*}\left(\frac{\boldsymbol{k}_1(\tau)}{\tau}\tau\right)}{\tau}\\ &= \hat{T}\left(\boldsymbol{\lambda}\right) \text{ by } (12)\end{aligned}\tag{54}$$

We will show at the end of the proof that the sequence $\bar{T}^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)/\tau$, $\tau=1,...$ is uniformly integrable, which will imply that

$$\begin{aligned}\limsup_{\tau\to\infty}\frac{\mathbb{E}\left[T_2\left|T_1=\tau,\ s_n=s\right.\right]}{\tau} &\leq \lim_{\tau\to\infty}\mathbb{E}\left[\frac{\bar{T}^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)}{\tau}\right]+\epsilon \quad \text{by } (52),\ (53)\\ &= \mathbb{E}\left[\lim_{\tau\to\infty}\frac{\bar{T}^{*}\left(\boldsymbol{k}_1\left(\tau\right)\right)}{\tau}\right]+\epsilon \text{ by uniform integrability}\\ &= \hat{T}\left(\boldsymbol{\lambda}\right)+\epsilon \text{ by } (54).\end{aligned}$$

Therefore, for δ such that $0<\delta<1-\hat{T}\left(\boldsymbol{\lambda}\right)-\epsilon$, there exists τ_δ such that for all pairs (τ,s) with $\tau>\tau_\delta$ it holds,

$$\mathbb{E}\left[T_2\left|T_1=\tau,\ S_1=s\right.\right]\leq(1-\delta)\,\tau,$$

hence,

$$\mathbb{E}\left[T_2\left|T_1=\tau,\ S_1=s\right.\right]\leq U+(1-\delta)\,\tau,$$

where

$$U=\max_{(\tau,s)\in\mathcal{G}:\tau\leq\tau_\delta}\mathbb{E}\left[T_2\left|T_1=\tau,\ S_1=s\right.\right].$$

Also, (50) is satisfied since $\lim_{\tau\to\infty}v\left(\tau\right)=\tau=\infty$.

It remains to show that $\bar{T}^*(\boldsymbol{k}_1(\tau))/\tau,\ \tau=1,2...$ is uniformly integrable. Using (5) we have

$$0 \leq \frac{\bar{T}^*(\boldsymbol{k}_1(\tau))}{\tau} \leq C_1 \sum_{i=1}^{n} \frac{k_{1,i}(\tau)}{\tau} + C_0 \tag{55}$$

Now, we have with probability one,

$$\lim_{\tau\to\infty} \frac{k_{1,i}(\tau)}{\tau} = \lambda_i$$

On the other hand, since the length of an epoch T_1 is independent of the arrivals during this epoch, we have

$$\frac{\mathbb{E}[k_{1,i}(\tau)]}{\tau} = \frac{\lambda_i \tau}{\tau} = \lambda_i$$

Since the nonnegative sequences $\frac{k_i(\tau)}{\tau},\ i=1,2,..,n,\ \tau=1,2,...$ converge both with probability one and in expectation, they are uniformly integrable (see Theorem 16.4 in [5]). Using this fact, uniform integrability of $\bar{T}^*(\boldsymbol{k}_1(\tau))/\tau$ follows from (55). ■

We next present the main theorem of this section, which shows the stability of policy π_ϵ.

Theorem 25. *For any $\boldsymbol{\lambda} \geq \mathbf{0}$ such that*

$$\hat{T}(\boldsymbol{\lambda}) < 1, \tag{56}$$

policy π_ϵ stabilizes the system.

Proof: The idea of the proof is the following. Assume that the system starts at time $t=0$ in system state s, with $\boldsymbol{A}(0)=\boldsymbol{k}$ packets at the inputs. We use the queue occupancy notation of $\boldsymbol{Q}_s^\pi(t),\ Q_s^\pi(t)$ from Section IV, but we henceforth omit the indices s and π to simplify the notation. Under π_ϵ, it will be shown through Theorem 22 that (56) implies that we can identify a state (τ_a, s_a) to which the chain $(T_m, S_m)_{m=1}^\infty$ returns infinitely often. Define $m_l,\ l=1,....,$ to be the sequence of epoch indices when the Markov chain is in state (τ_a, s_a). Then, due to the Markov property, the process consisting of the successive intervals between the times at which the process $(T_m, S_m)_{m=1}^\infty$ returns to state (τ_a, s_a), i.e.,

$$L_l = \sum_{j=m_l+1}^{m_{l+1}} T_j,\ l=1,2,... \tag{57}$$

consists of i.i.d. random variables and, as will be seen,

$$\mathbb{E}[L_l] < \infty. \tag{58}$$

Hence, the process

$$Z_0 = \sum_{j=1}^{m_1} T_j,\ Z_l = Z_{l-1} + L_l,\ l \geq 1,$$

constitutes a (delayed) renewal process.

Observe next that by the operation of π_ϵ, $\{\boldsymbol{Q}(Z_l)\}_{l=0}^\infty$, the number of packets in system at times Z_l, is statistically the same as the number of arrivals in a interval of length τ_a. Since packet arrivals are i.i.d and the operations of the process during the interval τ_a do not depend on these arrivals, $\{\boldsymbol{Q}(Z_l)\}_{l=0}^\infty$, consists of i.i.d. random variables with $\mathbb{E}[\boldsymbol{Q}(Z_l)] = \boldsymbol{\lambda}\tau_a < \infty$. This, and the operation of π_ϵ imply that the process $\{\boldsymbol{Q}(t)\}_{t=0}^\infty$, is regenerative with respect to $\{Z_l\}_{l=0}^\infty$. Let g be the period of the distribution of the cycle length L_l. It then follows from Corollary 1.5 p.128 in [12] and (58) that

$$\begin{aligned}\lim_{\alpha\to\infty} \frac{1}{g} \sum_{\beta=0}^{g-1} \Pr(Q(\alpha g+\beta) > q) &= \lim_{\alpha\to\infty} \frac{1}{g} \sum_{\beta=0}^{g-1} \mathbb{E}\left[1_{\{Q(\alpha g+\beta)>q\}}\right] \\ &= \frac{\mathbb{E}\left[\sum_{j=0}^{L_1-1} 1_{\{Q(Z_0+j)>q\}}\right]}{\mathbb{E}[L_1]}\end{aligned} \tag{59}$$

Observe next that the random variables $Y_j(q) = 1_{\{Q(Z_0+j)>q\}}$ are decreasing in q, and since $Q(Z_0+j)$ are finite, $\lim_{q\to\infty} 1_{\{Q(Z_0+j)>q\}} = 0$. Using the monotone convergence theorem we then have,

$$\begin{aligned}\lim_{q\to\infty} \mathbb{E}\left[\sum_{j=0}^{L_1-1} 1_{\{Q(Z_0+j)>q\}}\right] &= \mathbb{E}\left[\lim_{q\to\infty} \sum_{j=0}^{L_1-1} 1_{\{Q(Z_0+j)>q\}}\right] \\ &= 0.\end{aligned} \tag{60}$$

From (59), (60) we conclude that

$$\lim_{q\to\infty}\lim_{alpha\to\infty}\sum_{\beta=0}^{g-1}\Pr\left(Q\left(\alpha g+\beta\right)>q\right)=0 \tag{61}$$

Since for $t=\alpha_t g+\beta_t$ it holds

$$\Pr\left(Q\left(t\right)>q\right)\leq\sum_{\beta=0}^{g-1}\Pr\left(Q\left(\alpha_t g+\beta\right)>q\right)$$

we conclude from (61) that

$$\lim_{q\to\infty}\limsup_{t\to\infty}\Pr\left(Q\left(t\right)>q\right)=0$$

i.e., policy π_ϵ is stable.

To implement the plan outlined above we must show the existence of a state to which the Markov chain returns infinitely often, as well as (61). For this we will use Theorem 22 but because of the generality of the model under consideration, we cannot apriori claim irreducibility and aperiodicity in order to apply it directly. Instead, we rely first on Lemma 23 using the result of Lemma 24.

Let $C\left(\left(\tau,s\right)\right)$ be the communicating class to which a state $\left(\tau,s\right)$ belongs. We consider two cases as follows.

a) If $C\left(\left(\tau,s\right)\right)$ is essential, and $\left(T_1,\ S_1\right)=\left(\tau,s\right)$, we have $\left(T_m,\ S_m\right)\in C\left(\left(\tau,s\right)\right)\ m=1,2,...$ and the evolution of the process with initial condition $\left(\tau,s\right)$ constitutes an irreducible Markov Chain. If this chain is periodic with period d, then the process $\left\{\left(T_{dk+1},\ S_{dk+1}\right)\right\}_{k=0}^{\infty}$ is an aperiodic Markov Chain, [9] page 14. For this chain, we can apply Theorem 22 to show positive recurrence, as follows. Since by Lemma 24 the process $\left\{\left(T_m,S_m\right)\right\}_{m=1}^{\infty}$ satisfies (49), it also satisfies (51). Hence the process $\left\{\left(T_{dk+1},\ S_{dk+1}\right)\right\}_{k=0}^{\infty}$ satisfies (49) and since $\lim_{\tau\to\infty}v\left(\left(\tau,s\right)\right)=\infty$, it also satisfies (50). Therefore, by Lemma 23 we can apply Theorem 22 to $\left\{\left(T_{dk+1},\ S_{dk+1}\right)\right\}_{k=0}^{\infty}$ to deduce that it is geometrically ergodic with

$$\mathbb{E}\left[v\left(\widehat{X}\right)\right]=\mathbb{E}\left[\widehat{T}\right]<\infty. \tag{62}$$

From the above discussion we conclude that the initial state $\left(\tau,s\right)$ is visited infinitely often, and the successive visit indices are of the form $m_l=dV_l+1,\ l=0,1,2...,\ V_1=0$, where $V_l,\ l=1,...$ are integer valued i.i.d. random variables with

$$\mathbb{E}\left[V_l\right]<\infty \tag{63}$$

Let now,

$$\tilde{L}_k=\sum_{j=1}^{d}T_{dk+j}\ k=0,1,... \tag{64}$$

The nonnegative process $\left\{\tilde{L}_k\right\}_{k=0}^{\infty}$ is regenerative with respect to $\left\{V_l\right\}_{l=1}^{\infty}$ and by the regenerative theorem and (63) it holds,

$$\lim_{k\to\infty}\frac{\mathbb{E}\left[\sum_{m=0}^{k}\tilde{L}_m\right]}{k}=\frac{\mathbb{E}\left[\sum_{m=0}^{V_1-1}\tilde{L}_m\right]}{\mathbb{E}\left[V_1\right]}. \tag{65}$$

Observe next that by (57) and (64),

$$\mathbb{E}\left[\sum_{m=0}^{V_1-1}\tilde{L}_m\right]=\mathbb{E}\left[L_1\right] \tag{66}$$

Hence in order to show (58) it suffices to show

$$\mathbb{E}\left[\sum_{m=0}^{V_1-1}\tilde{L}_m\right]<\infty, \tag{67}$$

or, by (65),

$$\lim_{k\to\infty}\frac{\mathbb{E}\left[\sum_{m=0}^{k}\tilde{L}_m\right]}{k}<\infty. \tag{68}$$

Notice that by (64) we have,

$$\begin{aligned}\mathbb{E}\left[\sum_{m=0}^{k}\tilde{L}_m\right] &= \mathbb{E}\left[\sum_{m=0}^{d(k+1)}T_m\right]\\ &= \sum_{m=0}^{d(k+1)}\mathbb{E}\left[T_m\right]\\ &\le \sum_{m=0}^{d(k+1)}\left(\frac{U}{\delta}+(1-\delta)^m\,\tau\right)\ \text{by (51)}\\ &\le \frac{U}{\delta}\left(dk+d+1\right)\right)+\frac{\tau}{\delta}\end{aligned}$$

from which (68) follows.

b) Consider next the case where $C\left((\tau,s)\right)$ is inessential, i.e., there is at least one state $y\in\mathcal{G}-C\left((\tau,s)\right)$ such that for $x\in C\left((\tau,s)\right),\ x\rightsquigarrow y$ but $y\not\rightsquigarrow x$; here, with $x,\ y$ we denote pairs of the form (τ,s). Hence there is at least one other communicating class reachable from $C\left((\tau,s)\right)$. The communicating classes reachable from $C\left((\tau,s)\right)$ will be either essential or inessential. We argue that the process $\left\{(T_m,\ S_m)\right\}_{m=1}^{\infty}$ will enter an essential class in finite time. Assume the contrary, that is, there is a set of sample paths Ω_I with $\Pr\left\{\Omega_I\right\}>0$, for which the process remains always in some inessential class. Since inessential states are nonrecurrent (see [9], Theorem 4, p. 18) the process visits each inessential state only a finite number of times. This implies that on Ω_I, $\lim_{m\to\infty}T_m=\infty$, and since $\Pr\left\{\Omega_l\right\}>0$, we conclude that

$$\mathbb{E}\left[\lim_{m\to\infty}T_m\left|(T_1,S_1)=(\tau,s)\right.\right]=\infty.$$

Applying next Fatou's Lemma we have

$$\begin{aligned}\liminf_{m\to\infty}\mathbb{E}\left[T_m\left|(T_1,S_1)=(\tau,s)\right.\right] &\ge \mathbb{E}\left[\lim_{m\to\infty}T_m\left|(T_1,S_1)=(\tau,s)\right.\right]\\ &=\infty,\end{aligned}$$

which contradicts (51).

Since the process enters again an essential class in finite time, we can apply the arguments of case a) to complete the proof.

■